Mathematics 2200H – Mathematical Reasoning

TRENT UNIVERSITY, Fall 2021

Assignment #9

Embedding countable linear orders into $(\mathbb{Q}, <_{\mathbb{Q}})$

Due on Friday, 19 November.

*May be submitted on paper or via Blackboard.**

Recall from class that a *(strict) linear order*, let's denote it by $\triangleleft$, on a set A is a binary relation on A satisfying the following conditions:

i. $\triangleleft$ is *irreflexive*, *i.e.* for all $a \in A$ is not the case that $a \triangleleft a$.

ii. $\triangleleft$ is *transitive*, *i.e.* for all a, b, $c \in A$, if $a \triangleleft b$ and $b \triangleleft c$, then $a \triangleleft c$.

iii. $\triangleleft$ is *trichotomous*, *i.e.* for all a, $b \in A$, exactly one of $a \triangleleft b$, $a = b$, or $b \triangleleft a$ is true.

We showed in class that the linear order on the rational numbers, $<_{\mathbb{Q}}$, is indeed a linear order and that it also has the property of being *dense* in itself: between any two rational numbers there is another rational number, *i.e.* for all p, $q \in \mathbb{Q}$ with $p <_{\mathbb{Q}} q$ there is an $r \in \mathbb{Q}$ such that $a <_{\mathbb{Q}} r$ and $r <_{\mathbb{Q}} q$. It also has another property:

1. Show that $\mathbb{Q}$ has no endpoints. *i.e.* there are no least or greatest elements in $\mathbb{Q}$ as a whole when using the linear order $<_{\mathbb{Q}}$. *[3]*

A linear order $(A, <_A)$ can be *embedded* into a linear order $(B, <_B)$ if there is a function $f : A \to B$ that preserves order, *i.e.* for all x, $y \in A$, if $x <_A y$, then $f(x) <_B f(y)$. (Note that since we are dealing with strict linear orders, this means that f must be 1–1.) For example, we can embed the linear order of the natural numbers into the linear order of the rational numbers via the function $g(n) = n$.† Note that this is not the only way to pull off such an embedding. For example, the function given by $h(0) = 1$ and $h(n) = 3 - \frac{1}{n}$ also embeds the linear order of the natural numbers into the linear order of the rational numbers.

2. Show that every finite linear order can be embedded into the linear order of the rational numbers, *i.e.* into $(\mathbb{Q}, <_{\mathbb{Q}})$. *[2]*

Anticipating something we'll see a bit more of soon, a set A is said to be *countable* or *countably infinite* if there is a function $f : \mathbb{N} \to A$ which is 1–1 and onto. A little more informally, this means that we can list all of the elements of A in a list indexed by the natural numbers. For example, $\mathbb{Z}$ is a countable set since we can put all of its elements into such a list, with z_n denoting the integer indexed by n:

n	0	1	2	3	4	5	6	$\cdots$
z_n	0	1	-1	2	-2	3	-3	$\cdots$

You may amuse yourself by trying to define the corresponding function via a formula ... :-)

3. Show that every countable linear order can be embedded into the linear order of the rational numbers. *[5]*

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* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

† Technically, it ought to be $g(n) = \left[\left([(n,0)]_\sim, [(1,0)]_\sim\right)\right]_\approx$, but after seeing that, I trust everyone can see why we'd prefer to be a bit more informal.