Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2021

Assignment #4
"The Ramans do everything in threes."[†]
Due on Friday, 8 October.
May be submitted on paper or via Blackboard.[*]

Let's define a *set of threes* to be a set all of whose elements are sets which have exactly three elements. The object of this assignment is to devise a formula φ with one free variable, say x_0, in a pretty minimalist language for set theory, such that φ is true exactly when x is a set of threes. (Recall that an occurrence of a variable in a formula is free if it is not in the scope of a quantifier in that formula.) Here is a formal definition of the first-order language for set theory φ should be a formula of:

The symbols of the language are as follows:

Variables: x_0, x_1, x_2, ...
Connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$
Quantifiers: $\forall$, $\exists$
Parentheses: (,)
Equality: $=$
Set Membership: $\in$ (a 2-place relation)
Just to be paranoid: all of the above symbols are distinct, none is a substring of any other, and there are no other symbols in the language.

Note that the only terms of this language are the variables, as there are no constant symbols (not even for the empty set) or function symbols.

The formulas of the language are defined as follows:

1. For any variables x_i and x_j of the language, $(x_i = x_j)$ and $(x_i \in x_j)$ are formulas of the language.
2. If φ and ψ are any formulas of the language, then $(\neg\varphi)$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are also formulas of the language.
3. If φ is any formula of the language and x_i is any variable of the language, then $(\forall x_i \varphi)$ and $(\exists x_i \varphi)$ are also formulas of the language.
4. No string of symbols of the language is a formula of the language unless it was formed using (possibly many applications of) rules 1–3 above.

This language is inefficient in many ways, as it lacks most of the specialized symbols normally used in set theory, but at least it is uncomplicated as first-order languages go.

1. Give a formula ψ in the language specified above that has x_1 as its only free variable and is true exactly when x_1 has exactly three elements. *[5]*

2. Give a formula φ in the language specified above that has x_0 as its only free variable and is true exactly when x_0 is a set of threes. *[5]*

[†] From the end of Arthur C. Clarke's novel *Rendezvous with Rama*.

[*] All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`