Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2021

Assignment #3
A Little Number Theory
Due on Friday, 1 October.
*May be submitted on paper or via Blackboard.**

For this assignment you may assume that basic arithmetic on the integers works in the ways we are familiar with. (We will be showing that it does after we officially define the natural numbers and the integers in class.) You may also assume the following fact:

($\downarrow$) *(Descending Chain Condition)* Every strictly decreasing sequence of positive integers is finite.

That is, if you have a sequence of positive integers $a_0 > a_1 > a_2 > \cdots$, then it cannot be infinite. (In fact, it can have at most $|a_0|$ elements. Why?) This fact is surprisingly powerful; it turns out to be equivalent to being able to do induction, which we will see entirely too much of in the course of building up the various common number systems.

1. Suppose that a and b are positive integers with $a < b$. Use **($\downarrow$)** to show that there exist unique integers $c \geq 1$ and $0 \leq r < a$ such that $b = ca + r$. *[3]*

Recall that an integer a divides an integer b, often written as $a|b$, if $b = ca$ for some integer c, *i.e.* $r = 0$ above. [In the case where a and b are positive, anyway.] The *greatest common divisor* of two integers a and b, often written as $\gcd(a, b)$ or just (a, b), is the largest positive integer d such that $d|a$ and $d|b$.

2. Use **1** and ($\downarrow$) to show that any two positive integers do have a greatest common divisor. **[5]**

Hint: If you have trouble getting started, look up the Euclidean algorithm.

3. Suppose that a and b are positive integers and $d = (a, b)$ is their greatest common divisor. Show that there exist integers x and y (which need not be positive) such that $ax + by = d$. *[3]*

Hint: Trace the argument you did for **2** backwards.

* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`