Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2021

Assignment #11
The least uncountable ordinal
Due on Friday, 3 December.
May be submitted on paper or via Blackboard.[*]

Recall from class that an *ordinal* or *ordinal number* is a set α such that:

i. α is well-ordered by $\in$, *i.e.* $\in$ linear orders α and every non-empty subset A of α has an $\in$-least element.
ii. α is downward-closed under $\in$, *i.e.* whenever $b \in a$ and $a \in \alpha$, $b \in \alpha$ too.

Recall also that a set A is *finite* if there is a function $f : n \to A$ which is 1–1 and onto for some $n \in \mathbb{N}$, and is *countable* or *countably infinite* if there is a function $f : \mathbb{N} \to A$ which is 1–1 and onto. Note that one way to describe the set of natural numbers $\mathbb{N}$ is that it is the set of finite ordinals.

Let $\omega_1 = \{\, \alpha \mid \alpha \text{ is an ordinal which is finite or countable} \,\}$. Note that, notwithstanding the notation, this definition does not guarantee ω_1 is a set; after all, the collection of all ordinals, $ON = \{\, \alpha \mid \alpha \text{ is an ordinal} \,\}$, which has a similar definition, was shown in class not to be a set.

Do *one* (1) of the following problems.

1. Show that ω_1 is indeed a set. *[10]*

2. Assuming that ω_1 is indeed a set, show that ω_1 is an ordinal, but that it is not finite or countable. *[10]*

[*] All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`