Mathematics 2200H – Mathematical Reasoning

TRENT UNIVERSITY, Fall 2021

Assignment #10

Suprema and Infima

Due on Friday, 26 November.

*May be submitted on paper or via Blackboard.**

Recall from class that a *schnitt*, or *Dedeking cut*, is a set S of rational numbers satisfying the following conditions:

i. $S \neq \emptyset$ and $S \neq \mathbb{Q}$.

ii. S is *downward closed*, *i.e.* if $q \in \mathbb{Q}$, $p \in S$, and $q < p$, then $q \in S$.

iii. S has no largest element, *i.e.* for all $t \in S$, there is is a $u \in S$ such that $t < u$.

Officially, the real numbers are schnitts, *i.e.* $\mathbb{R} = \{ r \mid r \text{ is a schnitt} \}$. Informally, we should have pretty good intuition about the real numbers already, and the definition above amounts to having the real number r be represented by the schnitt $\{ q \in \mathbb{Q} \mid q < r \}$. We then defined the linear order on the real numbers, defined as schnitts, by $r <_{\mathbb{R}} s \iff r \subsetneq s$.

1. Suppose $T \subset \mathbb{R}$ is a set of real numbers with an upper bound $u \in \mathbb{R}$, *i.e.* for all $t \in T$, $t <_{\mathbb{R}} u$. Show that T has a least upper bound (often called a *supremum*) in $\mathbb{R}$. *[5]*

HINT: The proof done in class for the increasing sequence version of the Monotone Convergence Theorem didn't really require the sequence to be a sequence as such …

The twin to the result in question **1** is the one in question **2**, but it is a slightly harder to prove.

2. Suppose $T \subset \mathbb{R}$ is a set of real numbers with a lower bound $\ell \in \mathbb{R}$, *i.e.* for all $t \in T$, $\ell <_{\mathbb{R}} t$. Show that T has a greatest lower bound (often called an *infimum*) in $\mathbb{R}$. *[5]*

HINT: You can probably adapt most of the proof you used for question **1** by using a set operation other than union. There will likely be one small potential glitch you will have to deal with in showing that your greatest lower bound is a schnitt.

* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`