Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #$\mathbb{Q}\cap(-\infty, \mathbf{9})$
Cut to the $\mathbb{Q}$uick!
Due on Friday, 20 November.

Recall from various lectures that a *schnitt* or *Dedekind cut* is a subset S of $\mathbb{Q}$ such that:

1. $S \neq \emptyset$ and $S \neq \mathbb{Q}$.
2. S is "closed downwards": if $q \in S$ and $p \in \mathbb{Q}$ with $p < q$, then $p \in S$.
3. S has no largest element: if $q \in S$, then there is an $r \in S$ with $q < r$.

Intuitively, a schnitt S is $(-\infty, s) \cap \mathbb{Q}$ for some real number s. Formally, the real number in question *is* the schnitt S. The set of real numbers is then $\mathbb{R} = \{\, S \subset \mathbb{Q} \mid S \text{ is a schnitt} \,\}$. It is pretty easy to define particular real numbers as schnitts, *e.g.* $0_\mathbb{R} = \{\, q \in \mathbb{Q} \mid q < 0 \,\}$ and $1_\mathbb{R} = \{\, q \in \mathbb{Q} \mid q < 1 \,\}$, the operation of addition by $S + T = \{\, a + b \mid a \in S \text{ and } b \in T \,\}$, and the linear order on the reals by $S \leq T \iff S \subseteq T$.

1. Show that if S is a schnitt, then $S +_\mathbb{R} (-S) = 0_\mathbb{R}$. [Left unfinished in lecture ...] *[4]*

SOLUTION. Recall from lecture that if S is a schnitt, then

$$-S = \{\, -t \mid t \notin S \ \wedge \ t \neq \min(\mathbb{Q} \setminus S) \,\} \,,$$

where $\min(\mathbb{Q} \setminus S)$ is the least element of $\mathbb{Q} \setminus S$, assuming there is one. (Note that if a schnitt S represents a rational q, then $S = \{a \in \mathbb{Q} \mid a < q\}$ and $q = \min(\mathbb{Q} \setminus S)$, and otherwise $\mathbb{Q} \setminus S$ does not have a least element.) We will show that $S +_\mathbb{R} (-S) = 0_\mathbb{R}$ by showing that $S +_\mathbb{R} (-S) \subseteq 0_\mathbb{R}$ and $0_\mathbb{R} \subseteq S +_\mathbb{R} (-S)$.

First, suppose $x \in S +_\mathbb{R} (-S)$. By the definition of $+_\mathbb{R}$, this means that there are $a \in S$ and $b \in -S$ such that $x = a + b$. Since $b \in -S$, $b = -u$ for some $u \in \mathbb{Q} \setminus S$. As every element of $\mathbb{Q} \setminus S$ is greater than every element of S (since S is downward closed under $<$), it follows that $a < u$, and hence that $x = a + b = a + (-u) < 0$, so $x \in 0_\mathbb{R}$. Thus $S +_\mathbb{R} (-S) \subseteq 0_\mathbb{R}$.

Second, suppose $y \in 0_\mathbb{R}$, so $y < 0$. Choose a $u \in \mathbb{Q} \setminus S$ (other than the minimum element of $\mathbb{Q} \setminus S$, if there is one) such that $u + y \in S$. Then $-u \in -S$ by the definition of $-S$, so $y = y + 0 = y + u + (-u) = (u + y) + (-u) \in S +_\mathbb{R} (-S)$. Thus we also have $0_\mathbb{R} \subseteq S +_\mathbb{R} (-S)$, and so it follows that $S +_\mathbb{R} (-S) = 0_\mathbb{R}$.

The one thing that still needs some justification in the paragraph above is the ability to choose a $u \in \mathbb{Q} \setminus S$ (other than $\min(\mathbb{Q} \setminus S)$) such that $u + y \in S$. Intuitively, this borders on being obvious: S is a schnitt, so it consists of all the rationals up to some point and no rationals beyond that point, so we can find rationals below and above where S ends that are arbitrarily close to each other. All we then need to do is to find two rationals $w \in S$ and $u \in \mathbb{Q} \setminus S$ such that the distance $w - u$ between w and u is less that $|y|$. Then $u + y < u - (u - w) = w \in S$, so $u + y \in S$ by the downward closure property of schnitts.

We will verify the intuition that we can find rationals $w \in S$ and $u \in \mathbb{Q} \setminus S$ (other than $\min(\mathbb{Q} \setminus S)$) such that the distance $w - u$ between w and u is arbitrarily small. Suppose we are given an $\varepsilon > 0$, withe $\varepsilon \in \mathbb{Q}$. We will define sequences $\{w_n\}$ from S and $\{u_n\}$ from $\mathbb{Q} \setminus S$ inductively as follows:

- Choose a $w_0 \in S$ and a $u_0 \in \mathbb{Q} \setminus S$ arbitrarily. Note that since everything in S is less than everything in $\mathbb{Q} \setminus S$, we have $w_0 < u_0$.
- Given that $w_n \in S$ and $u_n \in \mathbb{Q} \setminus S$ have been defined, consider $v_n = \frac{1}{2}(w_n + u_n)$. If $v_n \in S$, let $w_{n+1} = v_n$ and let $u_{n+1} = u_n$, but if $v_n \in \mathbb{Q} \setminus S$, let $w_{n+1} = w_n$ and let $u_{n+1} = v_n$.

Note that we are allowing the possibility that w_n turns out to be the least element of $\mathbb{Q} \setminus S$, assuming there is one. If this should happen, then $w_k = w_n$ for all $k \geq n$. (Why?)

It is clear that this process will have $u_{n+1} - w_{n+1} = (u_n - w_n)/2$ for all $n \geq 0$, from which it is easy to deduce that $u_n - w_n = (u_0 - w_0)/2^n$ for all $n \geq 0$. For some n that is large enough, we will have $u_n - w_n = (u_0 - w_0)/2^n < \varepsilon/2$. If $u_n \neq \min(\mathbb{Q} \setminus S)$, we can take $u = u_n$ and $w = w_n$, giving us a $w \in S$ and $u \in \mathbb{Q} \setminus S$ that are less than $\varepsilon/2 < \varepsilon$ apart. If we are unlucky enough to have $u_n = \min(\mathbb{Q} \setminus S)$, we can take $u = u_n + \frac{\varepsilon}{2}$ and $w = w_n$, giving us giving us a $w \in S$ and $u \in \mathbb{Q} \setminus S$ that are less than $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ apart. Either way, we have what we need.

This completes the proof. Whew! ■

2. Define $\cdot_{\mathbb{R}}$, *i.e.* multiplication on the real numbers as defined via schnitts. [You need not develop the properties of multiplication, just define it fully.] *[6]*

Hint: First, define multiplication between two positive real numbers. Second, use what you did (and a bit of how you want multiplication to behave) to extend the definition to the cases where one or both of the numbers being multiplied is not positive.

SOLUTION. Following the hint, suppose S and T are schnitts, with $0_{\mathbb{R}} < S$ and $0_{\mathbb{R}} < T$. Note that this means that there are $p \in S$ with $0 < p$ and $q \in T$ with $0 < q$. In this case, let

$$S \cdot_{\mathbb{R}} T = \{\, q \in \mathbb{Q} \mid q \leq 0 \,\} \cup \{\, pq \mid p \in S \ \land \ q \in T \ \land \ 0 < p \ \land \ 0 < q \,\} \,.$$

It's not hard to check that this is a schnitt, but that wasn't asked for ... :-)

Having defined $\cdot_{\mathbb{R}}$ for positive reals, we can extend the definition to all other reals as follows:

If $0_{\mathbb{R}} < S$ and $T < 0_{\mathbb{R}}$, let $S \cdot_{\mathbb{R}} T = -(S \cdot_{\mathbb{R}} (-T))$.
If $S < 0_{\mathbb{R}}$ and $0_{\mathbb{R}} < T$, let $S \cdot_{\mathbb{R}} T = -((-S) \cdot_{\mathbb{R}} T)$.
If $S < 0_{\mathbb{R}}$ and $T < 0_{\mathbb{R}}$, let $S \cdot_{\mathbb{R}} T = (-S) \cdot_{\mathbb{R}} (-T)$.
If $S = 0_{\mathbb{R}}$ or $T = 0_{\mathbb{R}}$, let $S \cdot_{\mathbb{R}} T = 0_{\mathbb{R}}$.

Note that $S < 0_{\mathbb{R}}$ if and only if $0_{\mathbb{R}} < -S$, and that $0_{\mathbb{R}} < S$ if and only if $-S < 0_{\mathbb{R}}$, so the first three parts above use the definition of $\cdot_{\mathbb{R}}$ for positives.

That's that! Mind you, proving that multiplication so defined has the properties you expect of it, is, at best, rather tedious ... ■