Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #8
The Other Reals
Due on Friday, 13 November.

Let $\{a_n\}$ denote the sequence a_0, a_1, a_2, a_3, ... In this assignment all sequences will be sequences of rational numbers, *i.e.* every $a_k \in \mathbb{Q}$, indexed by the natural numbers.

DEFINITION. A sequence $\{a_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ [with $\varepsilon \in \mathbb{Q}$ for the sake of this assignment], there is an N_ε such that for all $m > n \geq N_\varepsilon$, we have $|a_m - a_n| < \varepsilon$.

Cauchy sequences are basically those sequences that ought to converge to some real number, if we had real numbers available to us. As a convenience – since you might not have had to remember this definition since first-year calculus – saying that a sequence converges means the following:

DEFINITION. A sequence $\{a_n\}$ *converges to* a (usually written as $a_n \to a$) or *has limit* a (usually written as $\lim\limits_{n\to\infty} a_n = a$) if for every $\varepsilon > 0$, there is an N_ε such that for all $n \geq N_\varepsilon$, we have $|a_n - a| < \varepsilon$.

1. Suppose $\{a_n\}$ is a convergent sequence, *i.e.* $a_n \to a$ for some a. Show that $\{a_n\}$ is a Cauchy sequence. *[5]*

SOLUTION. The basic idea is that if the terms of a sequence are all getting close to some number, then they must be getting close to each other too. Sadly, getting the details straight will require wading through some epsilonics.

Suppose $a_n \to a$ for some number a, and suppose we are given an $\varepsilon > 0$. To show that $\{a_n\}$ is a Cauchy sequence, we need to show that there is an N_ε such that if $m > n \geq N_\varepsilon$, then $|a_m - a_n| < \varepsilon$.

Since $\varepsilon > 0$, $\alpha = \varepsilon/2 > 0$ too, so, by the definition of $a_n \to a$, there is an N_α such that for all $n \geq N_\alpha$ we have $|a_n - a| < \alpha$. Now if $m > n \geq N_\alpha$, then

$$\begin{aligned} |a_m - a_n| &= |a_m - a + a - a_n| \\ &\leq |a_m - a| + |a - a_n| \\ &= |a_m - a| + |a_n - a| \\ &< \alpha + \alpha = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,, \end{aligned}$$

so taking $N_\varepsilon = N_\alpha$ lets us satisfy the definition of $\{a_n\}$ being a Cauchy sequence. $\square$

We can define an equivalence relation $\equiv$ on Cauchy sequences as follows:

DEFINITION. Suppose $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences. Then the two sequences are equivalent, $\{a_n\} \equiv \{b_n\}$, if and only if for every $\varepsilon > 0$ [with $\varepsilon \in \mathbb{Q}$], there is an N_ε such that for all $n \geq N_\varepsilon$, we have $|a_n - b_n| < \varepsilon$.

2. Verify that $\equiv$ is an equivalence relation on the set of Cauchy sequences of rational numbers. *[5]*

SOLUTION. We need to show that $\equiv$ is reflexive, commutative, and transitive.

i. $\equiv$ is reflexive: Suppose $\{a_n\}$ is a Cauchy sequence of rational numbers and $\varepsilon > 0$. Let $N_\varepsilon = 0$. Then for all $n \geq N_\varepsilon$, we have $|a_n - a_n| = |0| = 0 < \varepsilon$. By the definition of $\equiv$, this means that $\{a_n\} \equiv \{a_n\}$. Thus $\equiv$ is reflexive.

ii. $\equiv$ is commutative: Suppose $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of rational numbers, and assume that $\{a_n\} \equiv \{b_n\}$. We need to show that $\{b_n\} \equiv \{a_n\}$, *i.e.* that for every $\varepsilon > 0$, there is an N_ε such that if $n \geq N_\varepsilon$, then $|b_n - a_n| < \varepsilon$.

Suppose that we are given an $\varepsilon > 0$. Since $\{a_n\} \equiv \{b_n\}$, there is an N_ε such that if $n \geq N_\varepsilon$, then $|a_n - b_n| < \varepsilon$. However, then if $n \geq N_\varepsilon$, we have $|b_n - a_n| = |a_n - b_n| < \varepsilon$. It follows that $\{b_n\} \equiv \{a_n\}$, as required for $\equiv$ to be commutative.

ii. $\equiv$ is transitive: Suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are Cauchy sequences of rational numbers, and assume that $\{a_n\} \equiv \{b_n\}$ and $\{b_n\} \equiv \{c_n\}$. We need to show that $\{a_n\} \equiv \{c_n\}$, *i.e.* that for every $\varepsilon > 0$, there is an N_ε such that if $n \geq N_\varepsilon$, then $|a_n - c_n| < \varepsilon$.

Suppose that we are given an $\varepsilon > 0$. Let $\alpha = \beta = \varepsilon/2$, so $\alpha > 0$ and $\beta > 0$ too. Since $\{a_n\} \equiv \{b_n\}$ and $\{b_n\} \equiv \{c_n\}$, there exist N_α and N_β such that if $n \geq N_\alpha$, then $|a_n - b_n| < \alpha$, and if $n \geq N_beta$, then $|b_n - c_n| < \beta$. Letting $N_\varepsilon = \max(N_\alpha, N_\beta)$, it follows that if $n \geq N_\varepsilon$, then we have both $|a_n - b_n| < \alpha$ and $|b_n - c_n| < \beta$. This means that if $n \geq N_\varepsilon$, then

$$\begin{aligned} |a_n - c_n| &= |a_n - b_n + b_n - c_n| \\ &\leq |a_n - b_n| + |b_n - c_n| \\ &< \alpha + \beta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

so $\{a_n\} \equiv \{b_n\}$, as required for $\equiv$ to be transitive.

Since it is reflexive, commutative, and transitive, $\equiv$ is an equivalence relation. ■

The major alternative to defining the real numbers using *schnitts* or *Dedekind cuts*, as we will do in the lectures, is to define them using equivalence classes of Cauchy sequences of rational numbers. $\equiv$ is the equivalence relation in question. This method of defining the real numbers makes it a little easier to define the basic arithmetic operations on the real numbers, at the cost of making it rather harder to define and work with the linear order on the real numbers.