Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #$2^2 + 2^1 + 2^0$
Cancellation
Due on Friday, 6 November.

1. Show that $+_{\mathbb{Z}}$ satisfies the cancellation law for addition, *i.e.* for all a, b, $c \in \mathbb{Z}$, $a +_{\mathbb{Z}} c = b +_{\mathbb{Z}} c$ implies that $a = b$. *[6]*

SOLUTION. Suppose a, b, $c \in \mathbb{Z}$ and $a +_{\mathbb{Z}} c = b +_{\mathbb{Z}} c$. Following the hint given after question **2**, we use the fact that there exists a $-c \in \mathbb{Z}$ such that $c +_{\mathbb{Z}} (-c) = 0_{\mathbb{Z}}$ (see page 4 of the lecture *The Integers II*). This lets us use the following chain of reasoning, with a little help from the associativity of $+_{\mathbb{Z}}$:

$$\begin{aligned} a +_{\mathbb{Z}} c = b +_{\mathbb{Z}} c &\implies (a +_{\mathbb{Z}} c) +_{\mathbb{Z}} (-c) = (b +_{\mathbb{Z}} c) +_{\mathbb{Z}} (-c) \\ &\implies a +_{\mathbb{Z}} (c +_{\mathbb{Z}} (-c)) = b +_{\mathbb{Z}} (c +_{\mathbb{Z}} (-c)) \\ &\implies a +_{\mathbb{Z}} 0_{\mathbb{Z}} = b +_{\mathbb{Z}} 0_{\mathbb{Z}} \\ &\implies a = b \end{aligned}$$

We also used the fact (again, see *The Integers II*) that $p +_{\mathbb{Z}} 0_{\mathbb{Z}} = p$ for all $p \in \mathbb{Z}$. □

2. Show that $\cdot_{\mathbb{Z}}$ satisfies the cancellation law for multiplication, *i.e.* for all a, b, $c \in \mathbb{Z}$ with $c \neq 0_{\mathbb{Z}}$, $a \cdot_{\mathbb{Z}} c = b \cdot_{\mathbb{Z}} c$ implies that $a = b$. *[4]*

Hint: Having negatives makes **1** pretty easy. **2** is comparatively hard; it's worth remembering that if $[(s,t)]_\sim \neq 0_{\mathbb{Z}} = [(0,0)]_\sim$, then we must have $s \neq t$, so either $s = t + S(k)$ or $t = s + S(k)$ for some $k \in \mathbb{N}$.

SOLUTION. This problem is a good deal harder than **1** because we do not have multiplicative inverses to help us in $\mathbb{Z}$. (The sole exceptions being $1_{\mathbb{Z}}$ and $-1_{\mathbb{Z}}$, each of which is its own multiplicative inverse.) This means we will have to dig deeper into the relevant definitions, as a look at the hint also suggests.

Suppose we have a, b, $c \in \mathbb{Z}$ with $c \neq 0_{\mathbb{Z}}$ and $a \cdot_{\mathbb{Z}} c = b \cdot_{\mathbb{Z}} c$. Since a, b, $c \in \mathbb{Z}$, each of them is, by definition, an equivalence class, say $a = [(u,v)]_\sim$, $b = [(x,y)]_\sim$, and $c = [(s,t)]_\sim$. Since $c \neq 0_{\mathbb{Z}}$, we may, *per* the hint, assume that $s \neq t$, so either $s = t + S(k)$ or $t = s + S(k)$ for some $k \in \mathbb{N}$.

Suppose now that we are in the case that $s = t + S(k)$ for some $k \in \mathbb{N}$. Unwinding a lot of definitions:

$$\begin{aligned} a \cdot_{\mathbb{Z}} c = b \cdot_{\mathbb{Z}} c &\implies [(u,v)]_\sim \cdot_{\mathbb{Z}} [(s,t)]_\sim = [(x,y)]_\sim \cdot_{\mathbb{Z}} [(s,t)]_\sim \\ &\implies [(us + vt, ut + vs)]_\sim = [(xs + yt, xt + ys)]_\sim \\ &\implies (us + vt, ut + vs) \sim (xs + yt, xt + ys) \\ &\implies (us + vt) + (xt + ys) = (ut + vs) + (xs + yt) \end{aligned}$$

Now we use the properties of arithmetic in $\mathbb{N}$:

$$
\begin{aligned}
& (us + vt) + (xt + ys) = (ut + vs) + (xs + yt) \\
\implies & (u\,(t + S(k)) + vt) + (xt + y\,(t + S(k))) \\
& \quad = (ut + v\,(t + S(k))) + (x\,(t + S(k)) + yt) \\
\implies & ut + uS(k) + vt + xt + yt + yS(k) \\
& \quad = ut + vt + vS(k) + xt + xS(k) + yt \\
\implies & uS(k) + yS(k) + ut + vt + xt + yt \\
& \quad = vS(k) + xS(k) + ut + vt + xt + yt \\
\implies & uS(k) + yS(k) = vS(k) + xS(k) \\
\implies & (u + y)S(k) = (v + x)S(k) \\
\implies & u + y = v + x
\end{aligned}
$$

Note, in particular, that by the associativity and commutativity of addition in $\mathbb{N}$ we can regroup and rearrange a sum any way we want, in this case to facilitate the use of the cancellation law for addition in $\mathbb{N}$. We finish up by rewinding definitions:

$$
\begin{aligned}
u + y = v + x \implies & (u, v) \sim (x, y) \\
\implies & [(u, v)]_{\sim} = [(x, y)]_{\sim} \\
\implies & a = b
\end{aligned}
$$

Whew! The case where $t = s + S(k)$ for some $k \in \mathbb{N}$ has a very similar proof. ■