Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #0 + 1 + 2 + 3
Skipping a few number systems ahead . . . :-)
Due on Friday, 23 October.

The *quaternions* are the number system after the complex numbers:

$$\mathbb{H} = \{\, a + bi + ci + dj \mid a,\ b,\ c,\ d \in \mathbb{R} \,\}$$

where + and $\cdot$ work as usual except for the special numbers i, j, and k, which satisfy the following relations:

$$\begin{array}{lll} i^2 = j^2 = k^2 = -1 & & \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

Note that you have multiple square roots of -1 and that multiplication is not always commutative in the quaternions.

1. Suppose $\mathbf{a} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$ are two vectors in $\mathbb{R}^3$ and that $(pi + qj + rk)(si + tj + uk) = a + bi + ci + dj$. Verify that $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$. What does the real number a represent in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$? *[5]*

SOLUTION. Let's multiply out $(pi + qj + rk)(si + tj + uk)$ and see what we get:

$$\begin{aligned}(pi + qj + rk)(si + tj + uk) &= psi^2 + ptij + puik + qsji + qtj^2 + qujk \\ &\qquad + rski + rtkj + ruk^2 \\ &= ps(-1) + ptk + pu(-j) + qs(-k) + qt(-1) + qui \\ &\qquad + rsj + rt(-i) + ru(-1) \\ &= (-ps - qt - ru) + (qu - rt)i + (-pu + rs)j + (pt - qs)k\end{aligned}$$

Thus if $(pi + qj + rk)(si + tj + uk) = a + bi + ci + dj$, then $a = -ps - qt - ru$, $b = qu - rt$, $c = rs - pu$, and $d = pt - qs$. Then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & q & r \\ s & t & u \end{vmatrix} = \begin{vmatrix} q & r \\ t & u \end{vmatrix} \mathbf{i} - \begin{vmatrix} p & r \\ s & u \end{vmatrix} \mathbf{j} + \begin{vmatrix} p & q \\ s & t \end{vmatrix} \mathbf{k} \\ &= (qu - rt)\mathbf{i} - (pu - rs)\mathbf{j} + (pt - qs)\mathbf{k} = (qu - rt)\mathbf{i} + (rs - pu)\mathbf{j} + (pt - qs)\mathbf{k} \\ &= \begin{bmatrix} qu - rt \\ rs - pu \\ pt - qs \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}, \text{ as required.}\end{aligned}$$

As for a, observe that $a = -ps - qt - ru = -(ps + qt + ru) = -\begin{bmatrix} p \\ q \\ r \end{bmatrix} \cdot \begin{bmatrix} s \\ t \\ u \end{bmatrix} = -\mathbf{a} \cdot \mathbf{b}$, *i.e.* a is the negative of the dot product of $\mathbf{a}$ and $\mathbf{b}$. $\square$

2. Suppose $h = a + bi + cj + dk \in \mathbb{H}$ and $h \neq 0$. Express $h^{-1} = \dfrac{1}{h}$ as a quaternion in terms of a, b, c, and d. *[5]*

SOLUTION. If $h = a+bi+cj+dk \neq 0$, then not all of a, b, c, d are 0, so $a^2+b^2+c^2+d^2 \neq 0$. We claim that

$$\begin{aligned} h^{-1} &= \frac{a}{a^2+b^2+c^2+d^2} - \frac{bi}{a^2+b^2+c^2+d^2} - \frac{cj}{a^2+b^2+c^2+d^2} - \frac{dk}{a^2+b^2+c^2+d^2} \\ &= \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2} = \frac{\bar{h}}{h\bar{h}}\,, \text{ where } \bar{h} = a - bi - cj - dk \text{ is the } \textit{conjugate} \text{ of } h. \end{aligned}$$

We can verify this by multiplying out $h\bar{h} = (a+bi+cj+dk)\,(a-bi-cj-dk)$:

$$\begin{aligned} h\bar{h} &= (a+bi+cj+dk)\,(a-bi-cj-dk) \\ &= a^2 - abi - acj - adk + bai - b^2i^2 - bcij - bdik \\ &\qquad + caj - cbji - c^2j^2 - cdjk + dak - dbki - dckj - d^2k^2 \\ &= a^2 - abi - acj - adk + abi - b^2(-1) - bck - bd(-j) \\ &\qquad + acj - bc(-k) - c^2(-1) - cdi + adk - bdj - cd(-i) - d^2(-1) \\ &= a^2 - abi - acj - adk + abi + b^2 - bck + bdj \\ &\qquad + acj + bck + c^2 - cdi + adk - bdj + cdi + d^2 \\ &= \left(a^2+b^2+c^2+d^2\right) + (-ab+ab-cd+cd)i \\ &\qquad + (-ac+bd+ac-bd)j + (-ad-bc+bc+ad)k \\ &= \left(a^2+b^2+c^2+d^2\right) + 0i + 0j + 0k = a^2+b^2+c^2+d^2 \qquad \blacksquare \end{aligned}$$