Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #$\mathbf{2^2+1}$
Exponentiation in $\mathbb{N}$
Due on Friday, 16 October.

Let's define the operation of exponentiation in the natural numbers as follows:

- For all $n \in \mathbb{N}$, let $n^0 = 1$.
- Given that n^k has been defined for some $k \in \mathbb{N}$ and all $n \in \mathbb{N}$, let $n^{S(k)} = \left(n^k\right) \cdot n$.

In answering the questions below, you may use the definitions and all the properties of $+$ and $\cdot$ on $\mathbb{N}$ developed in the lectures, plus the (augmented) Peano axioms we are using, plus the definition of exponentiation above.

1. Prove that $0^k = 0$ for all $k \geq 0$. *[2]*

SOLUTION. This isn't quite true since, by the definition of exponentiation given above, $0^0 = 1$. However, it is true that $0^k = 0$ for all $k \geq 1$, which we prove by induction on k:

Base Step: ($k = 1$) $0^1 = 0^{S(0)} = \left(0^0\right) \cdot 0 = 1 \cdot 0 = 0$ by the definitions of exponentiation and multiplication.

Inductive Hypothesis: For some $k \geq 1$, we have $0^k = 0$.

Inductive Step: ($k \to k+1$) Assume the Inductive Hypothesis. Then $0^{k+1} = 0^{S(k)} = \left(0^k\right) \cdot 0 = 0 \cdot 0 = 0$ by the definitions of exponentiation and multiplication.

Therefore $0^k = 0$ for all $k \geq 1$ by induction. $\square$

2. Prove that for all n, m, $k \in \mathbb{N}$, $(n^m) \cdot \left(n^k\right) = n^{m+k}$. *[4]*

SOLUTION. We proceed by induction on k:

Base Step: ($k = 0$) For all n, $m \in \mathbb{N}$, $(n^m) \cdot \left(n^0\right) = (n^m) \cdot 1 = n^m = n^{m+0}$, using the definition of exponentiation and the basic properties of multiplication and addition.

Inductive Hypothesis: For some $k \geq 1$ and all n, $m \in \mathbb{N}$, we have $(n^m) \cdot \left(n^k\right) = n^{m+k}$.

Inductive Step: ($k \to k+1$) Assume the Inductive Hypothesis and suppose n, $m \in \mathbb{N}$. Then

$$\begin{aligned}(n^m) \cdot \left(n^{k+1}\right) &= (n^m) \cdot \left(n^{S(k)}\right) = (n^m) \cdot \left(\left(n^k\right) \cdot n\right) = \left((n^m) \cdot \left(n^k\right)\right) \cdot n \\ &= \left(n^{m+k}\right) \cdot n = n^{S(m+k)} = n^{(m+k)+1} = n^{m+(k+1)}\,,\end{aligned}$$

using the Inductive Hypothesis, the definition of exponentiation, and various properties of multiplication and addition, especially associativity.

Therefore $(n^m) \cdot \left(n^k\right) = n^{m+k}$ for all n, m, $k \in \mathbb{N}$ by induction. $\square$

3. Prove that for all n, m, $k \in \mathbb{N}$, $\left(n^k\right) \cdot \left(m^k\right) = (n \cdot m)^k$. *[4]*

Solution. We proceed by induction on k:

Base Step: ($k = 0$) For all n, $m \in \mathbb{N}$, $\left(n^0\right) \cdot \left(m^0\right) = 1 \cdot 1 = 1 = (n \cdot m)^0$, using the definition of exponentiation and the basic properties of multiplication.

Inductive Hypothesis: For some $k \geq 1$ and all n, $m \in \mathbb{N}$, we have $\left(n^k\right) \cdot \left(m^k\right) = (n \cdot m)^k$.

Inductive Step: ($k \to k+1$) Assume the Inductive Hypothesis and suppose n, $m \in \mathbb{N}$. Then

$$\begin{aligned}
\left(n^{k+1}\right) \cdot \left(m^{k+1}\right) &= \left(n^{S(k)}\right) \cdot \left(m^{S(k)}\right) = \left(\left(n^k\right) \cdot n\right) \cdot \left(\left(m^k\right) \cdot m\right) \\
&= \left(\left(\left(n^k\right) \cdot n\right) \cdot \left(m^k\right)\right) \cdot m = \left(\left(n^k\right) \cdot \left(n \cdot \left(m^k\right)\right)\right) \cdot m \\
&= \left(\left(n^k\right) \cdot \left(\left(m^k\right) \cdot n\right)\right) \cdot m = \left(\left(\left(n^k\right) \cdot \left(m^k\right)\right) \cdot n\right) \cdot m \\
&= \left(\left(n^k\right) \cdot \left(m^k\right)\right) \cdot (n \cdot m) = (n \cdot m)^k \cdot (n \cdot m) \\
&= (n \cdot m)^{S(k)} = (n \cdot m)^{k+1} \ ,
\end{aligned}$$

using the Inductive Hypothesis, the definition of exponentiation, and various properties of multiplication and addition, especially associativity of multiplication. (Notice how many of the steps above involve shuffling parentheses about using said associativity!)

Therefore $\left(n^k\right) \cdot \left(m^k\right) = (n \cdot m)^k$ for all n, m, $k \in \mathbb{N}$ by induction. ■