Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2020

Solutions to Assignment #2+2
An almost universe of set theory
Due on Friday, 9 October.

Let's define sets V_n for all $n \geq 0$ as follows:

$V_0 = \emptyset$
If V_n has been defined for some $n \geq 0$, let $V_{n+1} = \mathcal{P}(V_n) = \{ X \mid X \subseteq V_n \}$.

Thus

$$\begin{aligned} V_0 &= \emptyset \\ V_1 &= \mathcal{P}(V_0) = \mathcal{P}(\emptyset) = \{\emptyset\} \\ V_2 &= \mathcal{P}(V_1) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}(V_2) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

and so on. For anyone who noticed that $V_0 = \emptyset = 0$, $V_1 = S(0) = 1$, and $V_2 = S(1) = 2$, please note that this pattern stops at $n = 3$: $3 = S(2) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subsetneq \mathcal{P}(V_2) = V_3$. It is, however, true that for all $n \geq 0$, we have that 0, 1, 2, … , n are all elements of V_{n+1}, so it turns out that $n \subseteq V_n$ for all $n \geq 0$.

1. Give an inductive argument showing that $V_n \subseteq V_{n+1}$ for all $n \geq 0$. *[4]*

Solution. We will use induction on n to show that $V_n \subseteq V_{n+1}$ for all $n \geq 0$.

Base Step: ($n = 0$) By definition, $V_0 = \emptyset$. Since the empty set is a subset of every set, it follows that $V_0 \subseteq V_1$.

Inductive Hypothesis: For some $n \geq 0$, $V_n \subseteq V_{n+1}$.

Inductive Step: We need to show, assuming the Inductive Hypothesis, that $V_{n+1} \subseteq V_{n+2}$.

Suppose $x \in V_{n+1}$. Since $V_{n+1} = \mathcal{P}(V_n) = \{ x \mid x \subset V_n \}$ by definition, this means that $x \subseteq V_n$. By the Inductive Hypothesis, $V_n \subseteq V_{n+1}$, so $x \subseteq V_{n+1}$ too. It now follows that $x \in \mathcal{P}(V_{n+1}) = V_{n+2}$. Hence every x in V_{n+1} is also in V_{n+2}, *i.e.* $V_{n+1} \subseteq V_{n+2}$.

Thus $V_n \subseteq V_{n+1}$ for all $n \geq 0$ by induction. □

Note: It follows from **1** that if $n \leq k$, then $V_n \subseteq V_k$. A modification of the argument given above could be used to show this fact directly. Note also that it follows from the argument at the inductive step that if $x \in V_n$, then $x \subseteq V_n$. (This is vacuously true for $n = 0$.)

What happens when we put all of the V_ns together? To put it another way, what is the "limit" of the increasing sequence of sets $V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$? It's the following object:

$$V_\omega = \bigcup_{n=0}^{\infty} V_n = \{ x \mid x \in V_n \text{ for some } n \geq 0 \}$$

V_ω is an infinite set since $\mathbb{N} = \{0, 1, 2, 3, \dots\} \subseteq V_\omega$. (Why?) In fact, it is large and complex enough that most of the axioms of set theory would be true if the universe of sets were V_ω.

2. Which of the axioms of set theory, as developed in the set theory lectures I-IV, would be true if the universe of sets was V_ω? Explain why or why not for each. *[6]*

SOLUTION. We'll run through the axioms of set theory we have so far and see if they would be true if V_ω was the collection of all the sets there were.

0° *Empty Set Axiom.* This axiom asserts that there exists a set which has no elements. For this to be true in V_ω, there must be a set in V_ω which has no elements that are also in V_ω. The actual empty set is an element of $V_1 = \{\emptyset\}$, and hence of V_ω (by its definition), and $\emptyset$ has no elements that are in V_ω (or anywhere else :-).

Thus the Empty Set Axiom holds in V_ω.

1° *Pair Set Axiom.* This axiom asserts that if x and y are sets, then $\{x, y\}$ is also a set. For this to be true in V_ω, we we need to have that for all x, $y \in V_\omega$, we also have $\{x, y\} \in V_\omega$.

Suppose x, $y \in V_\omega$. By the definition of V_ω, there exist n, $m \in \mathbb{N}$ such that $x \in V_n$ and $y \in V_m$. Let $k = \max(n.m)$. It follows from **1** that $V_n \subseteq V_k$ and $V_m \subseteq V_k$, so x, $y \in V_k$. It follows that $\{x, y\} \subseteq V_k$, and so $\{x, y\} \in \mathcal{P}(V_k) = V_{k+1}$. By the definition of V_ω, it follows that $\{x, y\} \in V_\omega$.

Thus the Pair Set Axiom holds in V_ω.

2° *Union Axiom.* This axiom asserts that if x is a set, then the collection of all elements of elements of x, $\bigcup x = \bigcup_{y \in x} y = \{ z \mid \exists y \in x\,((z \in y) \wedge (y \in x)) \}$, is also a set. For this to be true in V_ω, we need to have that for all $x \in V_\omega$, we also have $\bigcup x \in V_\omega$.

Suppose $x \in V_\omega$. Then $x \in V_n$ for some $n \geq 1$ by the definition of V_ω. ($V_0 = \emptyset$, so $x \notin V_0 \ldots$) If $n = 1$, then $x \in V_1 = \{\emptyset\}$, so $x = \emptyset$. In this case $\bigcup x = \bigcup \emptyset = \emptyset \in V_\omega$. If $n \geq 2$, then $x \in V_n = \mathcal{P}(V_{n-1})$, so $x \subseteq V_{n-1}$. That is, $y \in V_{n-1} = \mathcal{P}(V_{n-2})$ for every $y \in x$, *i.e.* $y \subseteq V_{n-2}$ for every $y \in x$. It follows that $\bigcup x \subseteq V_{n-2}$, so $\bigcup x \in \mathcal{P}(V_{n-2}) = V_{n-1}$. Therefore, by the definition of V_ω, $\bigcup x \in V_\omega$ in this case too.

Thus the Union Axiom holds in V_ω.

3° *Power Set Axiom.* This axiom asserts that if x is a set, then $\mathcal{P}(x) = \{ y \mid y \subseteq x \}$ is also a set. For this to be true in V_ω we need to have that for all $x \in V_\omega$, we also have $\mathcal{P}(x) \in V_\omega$.

Suppose $x \in V_\omega$. Then $x \in V_n$ for some $n \geq 1$. (Again, $V_0 = \emptyset$, so $x \notin V_0$.) Since $V_n = \mathcal{P}(V_{n-1})$, it follows that $x \subseteq V_{n-1}$. This, in turn, means that if $y \subseteq x$, then $y \subseteq V_{n-1}$, and so $y \in \mathcal{P}(V_{n-1}) = V_n$. As $y \in V_n$ for every $y \subseteq x$, it follows that $\mathcal{P}(x) = \{ y \mid y \subseteq x \} \subseteq V_n$, and hence that $\mathcal{P}(x) \in \mathcal{P}(V_n) = V_{n+1}$. Therefore, by the definition of V_ω, we have $\mathcal{P}(x) \in V_\omega$.

Thus the Power Set Axiom holds in V_ω.

4° *Extension Axiom.* This axiom asserts that two sets are equal if they have the same elements. For this to be true in V_ω we need to have that for all x, $y \in V_\omega$, $x = y$ if for all $z \in V_\omega$, $z \in x$ if and only if $z \in y$.

As was noted after the main proof for **1** above, part of that proof showed that if $x \in V_n$ for some $n \in \mathbb{N}$, then $x \subseteq V_n$. Since any $x \in V_\omega$ is there because $x \in V_n$ for some n and since $V_n \subseteq V_\omega$, it follows that $x \subseteq V_\omega$ for any $x \in V_\omega$. Thus if x, $y \in V_omega$, we have that $z \in x$ if and only if $z \in y$ for all $z \in V_\omega$ actually boils down to $z \in x$ if and only if $z \in y$ for all z (since $x \subseteq V_\omega$ and $y \subseteq V_\omega$), so $x = y$.

Thus the Extension Axiom holds in V_ω.

5° *Foundation Axiom.* This axiom asserts that every non-empty set has an element with which the set has no elements in common, *i.e.* for every $x \neq \emptyset$ there is a $y \in x$ such that $y \cap x = \emptyset$. This is, in particular, true of all sets $x \in V_\omega$. As noted above, $x \subseteq V_\omega$ when $x \in V_\omega$, so a y in x with $y \cap x = \emptyset$ is also in V_ω.

Thus the Foundation Axiom holds in V_ω.

6° *Comprehension Axiom.* This axiom asserts that a definable subset of a set is also a set, *i.e.* if x is a set and $\varphi(z)$ is formula that is true or false depending on what set z is plugged into it, then $\{ z \in x \mid \varphi(z) \text{ is true} \}$ is also a set. For this to be true in V_ω such definable subsets of sets in V_ω would also have to be in V_ω.

Suppose $x \in V_\omega$ and $y = \{ z \in x \mid \varphi(z) \text{ is true} \}$ is a definable subset of x. Then $x \in V_n$ for some n and so, as previously noted, $x \subseteq V_n$ too. Since $y \subseteq x$, $y \subseteq V_\omega$ as well, so $y \in \mathcal{P}(V_n) = V_{n+1}$. By the definition of V_ω, it follows that $y \in V_\omega$.

Thus the Comprehension Axiom holds in V_ω.

To help deal with the Replacement and Infinity Axioms, we will first prove a small lemma:

LEMMA. V_n is finite for every $n \in \mathbb{N}$.

PROOF. We will proceed by induction on n.
Base Step. ($n = 0$) $V_0 = \emptyset$ has 0 elements, so it is finite.
Inductive Hypothesis. V_n is finite for some $n \geq 0$.
Inductive Step. We need to show that V_{n+1} is finite. By the Inductive Hypothesis, V_n is finite, say with k elements. Then $V_{n+1} = \mathcal{P}(V_n)$ has 2^k elements, because each subset of V_n is determined by making a choice for every one of the k elements of V_n of whether to include it in the subset or not. Thus V_{n+1} is also finite. //

7° *Replacement Axiom.* This axiom asserts that if we can define a function with a given domain, then the range of the function is also a set. More precisely, if A is a set and $\varphi(x, y)$ is a formula such that for every $x \in A$ there is exactly one set y making $\varphi(x, y)$ true, then $B = f''A = \{ y \mid \exists x \in A : \varphi(x, y) \text{ is true} \}$ is also a set. For this to be true in V_ω we need to have that if $A \in V_\omega$ and $\varphi(x, y)$ is a formula such that for every $x \in A$ there is exactly one set $y \in V_\omega$ making $\varphi(x, y)$ true, then $B = \{ y \mid \exists x \in A : \varphi(x, y) \text{ is true} \} \in V_\omega$.

Suppose that $A \in V_\omega$ and $\varphi(x, y)$ is a formula such that for every $x \in A$ there is exactly one set $y \in V_\omega$ making $\varphi(x, y)$ true. Note that each such $y \in V_\omega$ must be an element of V_n for some n by the definition of V_ω. Let $m = \max \{ n \in \mathbb{N} \mid \exists x \in A \exists y \, (\varphi(x, y) \wedge (y \in V_n)) \}$. This maximum is well defined because $A \in V_k$ for some k, so $A \subseteq V_k$, and V_k is finite by the Lemma, so A is finite. Then $B = \{ y \mid \exists x \in A : \varphi(x, y) \text{ is true} \} \subseteq V_m$, so $B \in \mathcal{P}(V_m) = V_{m+1}$. It follows by the definition of V_ω that $B \in V_\omega$.

Thus the Replacement Axiom holds in V_ω.

8° *Infinity Axiom.* This axiom asserts that the collection of all the natural numbers, $\mathbb{N}$, is a set. For this axiom to be true if the universe of sets was V_ω, we would need to have $\mathbb{N} \in V_\omega$. However, every set in V_ω is finite: using facts noted previously, if $x \in V_\omega$, then $x \in V_n$ for some n, and hence $x \subseteq V_n$. Since V_n is finite, x must also be finite. Since $\mathbb{N}$ is infinite, it cannot be an element of V_ω.

Thus the Infinity Axiom does *not* hold in V_ω. Note that although $\mathbb{N} \notin V_\omega$, it is true that every natural number n is an element of V_ω, *i.e.* $\mathbb{N} \subseteq V_\omega$.

Our final tally is that every one of the axioms we have looked at (the so-called Zermelo-Fraenkel (ZF) axioms for set theory), except for the Infinity Axiom, are true in V_ω. ■