Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #S(2)
A little bit of Aristotle
Due on Friday, 2 October.

The Greek philosopher Aristotle seems to have been the first to write about logic as a subject in its own right. His logic relied on a number of rules of procedures or argument forms whose uses are called "syllogisms", such as the following:

*All humans are featherless bipeds.**
Meredith is human.
Therefore Meredith is a featherless biped.

Suppose we have two one-place relations H and F in a first-order language, whose intended meanings are "x is human" for $H(x)$ and "x is a featherless biped" for $F(x)$, as well as a constant symbol m representing Meredith. Then the syllogism above claims that the assumptions $\forall x\,(H(x) \to F(x))$ and $H(m)$ imply the conclusion that $F(m)$.

1. Give a deduction, using the axiom schema given in class and with Modus Ponens as the only rule of procedure, of $F(m)$ from the premisses $\forall x\,(H(x) \to F(x))$ and $H(m)$. You may assume that m is substitutable for x in the formula $(H(x) \to F(x))$. *[2]*

SOLUTION. Note that since m is a constant, it is a term, and observe that if α is the formula $(H(x) \to F(x))$, then α^x_m (that is, α with x substituted for by m) is $(H(m) \to F(m))$. Off we go, routinely omitting the outermost parentheses for readability:

1.	$(\forall x\,(H(x) \to F(x))) \to (H(m) \to F(m))$	(A4)
2.	$\forall x\,(H(x) \to F(x))$	Premiss
3.	$H(m) \to F(m)$	1,2 MP
4.	$H(m)$	Premiss
5.	$F(m)$	3,4 MP

That's that! $\square$

2. Translate the three sentences

Some humans are featherless bipeds.
No humans are featherless bipeds.
Some humans are not featherless bipeds.

into suitable formulas of a first-order language including the relations H and F described above. *[3]*

SOLUTION. Here we go, once again omitting the outermost parentheses for readability:

Some humans are featherless bipeds.	$\exists x\,(H(x) \land F(x))$
No humans are featherless bipeds.	$\forall x\,(H(x) \to (\neg F(x)))$

* Aristotle apparently characterized humans as "featherless bipeds" in one of his works on natural philosophy.

Some humans are not featherless bipeds. $\exists x\,(H(x) \wedge (\neg F(x)))$

Of course, any logically equivalent formula to those given above would do as well in each case. □

3. Together with "All humans are featherless bipeds", the sentences in **2** reflect the four main sentence forms involving quantifiers studied in Aristotle's logic. If there are no humans, but there are some featherless bipeds, in the universe these four sentences are talking about, which of them must be true? If, instead, there are no featherless bipeds, but there are some humans in that universe, which of the four sentences must be true? (Do explain why in each case, but you need not give formal deductions.) *[3]*

SOLUTION. First, suppose that there are no humans, but there are some featherless bipeds, in the universe. That is, $H(x)$ is always false, but $F(x)$ is (at least) sometimes true. Then:

Since $H(x)$ is always false, $H(x) \to F(x)$ must always be true, so $\forall x\,(H(x) \to F(x))$ is true.

Since $H(x)$ is always false, $H(x) \wedge F(x)$ is always false, so $\exists x\,(H(x) \wedge F(x))$ is false.

Since $H(x)$ is always false, $H(x) \to (\neg F(x))$ is always true, so $\forall x\,(H(x) \to (\neg F(x)))$ is true.

Since $H(x)$ is always false, $H(x) \wedge (\neg F(x))$ is always false, so $\exists x\,(H(x) \wedge (\neg F(x)))$ is false.

Second, suppose there are no featherless bipeds, but there are some humans, in the universe. That is, $H(x)$ is at least sometimes true and $F(x)$ is always false. Then:

Since $H(x)$ is true and $F(x)$ is false for at least one value of x, $H(x) \to F(x)$ is false for at least one value of x, so $\forall x\,(H(x) \to F(x))$ is false.

Since $F(x)$ is always false, $H(x) \wedge F(x)$ is always false, so $\exists x\,(H(x) \wedge F(x))$ is false.

Since $F(x)$ is always false, $\neg F(x)$ is always true, so $H(x) \to (\neg F(x))$ is always true, so $\forall x\,(H(x) \to (\neg F(x)))$ is true.

Since $F(x)$ is always false, $\neg F(x)$ is always true, and since $H(x)$ is sometimes true, there is at least one value of x such that $H(x) \wedge (\neg F(x))$ is true, so $\exists x\,(H(x) \wedge (\neg F(x)))$ is true.

Done! ■

4. Aristotelian logic pretty much consists of propositional logic, plus just enough first-order logic to properly handle the four sentence forms given above. Give an example of a mathematical result and its proof that Aristotelian logic is *not* adequate to handle, and explain why this is so. *[2]*

SOLUTION. Any result with several quantifiers and a statement that ties together the variables these control is likely to work. For example, consider the following statement (with lots of parentheses omitted :-) about the natural numbers: $\forall x \forall y \exists z \exists w\,(x + w = y + z)$ It clearly follows from the commutativity of addition (*i.e.* $\forall x \forall y\,(x + y = y + x)$) for the natural numbers – just plug in y for w and x for z in the equation – but it's hard to see how to deduce it using Aristotelian syllogisms. Facts about limits, which require parsing the more complex quantifier structure of the ε–δ definition are even better candidates for being undoable in Aristotelian logic. ■