Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solution to Assignment #1011_2
Subsets and Functions
Due on Friday, 4 December.

Recall that the power set of a set A is $\mathcal{P}(A) = \{\, X \mid X \subseteq A \,\}$. The somewhat ugly notation ${}^{A}2$ denotes the set of functions from A to $2 = \{0, 1\}$, that is

$$ {}^{A}2 = \{\, f \mid f \text{ is a function } A \to \{0,1\} \,\} \,. $$

1. Show that $\left\| {}^{A}2 \right\| = \|\mathcal{P}(A)\|$ for any set $A \neq \emptyset$. *[10]*

Hint: Going back to the definition of cardinality, you need to find a suitable correspondence between functions $f : A \to \{0,1\}$ and subsets $X \subseteq A$.

SOLUTION. Define $\chi : {}^{A}2 \to \mathcal{P}(A)$ by $\chi(f) = \{\, a \in A \mid f(a) = 1 \,\}$. We claim that χ is 1–1 and onto.

Suppose f, $g \in {}^{A}2$ and $f \neq g$. Since the two functions $A \to 2$ are not equal, there is some $a \in A$ for which $f(a) \neq g(a)$. As each of $f(a)$ and $g(a)$ is either 0 or 1, this means that either $f(a) = 1$ and $g(a) = 0$ or $f(a) = 0$ and $g(a) = 1$. In the former case $a \in \chi(f)$ but $a \notin \chi(g)$ and in the latter case $a \in \chi g$ but $a \notin \chi(f)$. Either way, $\chi(f) \neq \chi(g)$. Thus χ is 1–1.

Suppose $X \in \mathcal{P}(A)$, *i.e.* $X \subseteq A$. Define $f : A \to 2$ by $f(a) = \begin{cases} 1 & a \in X \\ 0 & a \notin X \end{cases}$. It then follows from the definition of χ that $\chi(f) = \{\, a \in A \mid f(a) = 1 \,\} = \{\, a \in A \mid a \in X \,\} = X$. Thus χ is onto.

Since we have a 1–1 function $\chi : {}^{A}2 \to \mathcal{P}(A)$, we have $\left\| {}^{A}2 \right\| = \|\mathcal{P}(A)\|$ by the definition of cardinality. ■