Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2020
Solutions to Assignment #$S(S(S(S(S(S(S(S(S(S(0))))))))))$
A Little Ordinal Arithmetic
Due on Friday, 27 November.

Recall from lecture that if $\triangleleft$ is a well-order on a set A, then the *order type* of $(A, \triangleleft)$ is the unique ordinal α such that there is a 1–1 onto function $f : \alpha \to A$ such that for all γ, $\beta \in \alpha$, if $\gamma < \beta$, then $f(\gamma) \triangleleft f(\beta)$. We will use this basic idea to define addition and multiplication for ordinals as follows.

$+_o$: Suppose α and β are any two ordinals. Intuitively, $\alpha +_o \beta$ is the order type of putting a copy of β right after after everything in α. Formally, we define $\blacktriangleleft$ on $(\{0\} \times \alpha)) \cup (\{1\} \times \beta)$ by $(u, x) \blacktriangleleft (w, y)$ if either $u < w$ or $u = w$ and $x < y$. Then $\alpha +_o \beta$ is the order type of $((\{0\} \times \alpha)) \cup (\{1\} \times \beta), \blacktriangleleft)$.

$\cdot_o$: Suppose α and β are any two ordinals. Intuitively, $\alpha \cdot_o \beta$ is the order type of inserting a separate copy of all of α in place of each element of β. Formally, we define $\prec$ on $\alpha \times \beta$ by $(a, b) \prec (c, d)$ if either $b < d$ or $b = d$ and $a < c$. Then $\alpha \cdot_o \beta$ is the order type of $(\alpha \times \beta, \prec)$.

When you stick to finite ordinals, *i.e.* natural numbers, these definitions end up giving the same results as $+_{\mathbb{N}}$ and $\cdot_{\mathbb{N}}$. However, both operations behave differently when applied to infinite ordinals, as you will work out below.

Recall that we denote the first infinite ordinal, $\mathbb{N}$, by ω when we think of it as an ordinal in its own right rather than just as the set of natural numbers.

1. Show, in detail, that $1 +_o \omega = \omega < \omega + 1$. *[4]*

Solution. We claim that $1 +_o \omega = \omega$, but $\omega +_o 1 = S(\omega)$.

Following the hint and using the idea that $\alpha +_o \beta$ simply puts all of (a copy of) α before all of (a copy of) β, we can present the linear orders involved as follows:

$$\begin{aligned} 1 :&\quad \bullet \\ \omega :&\quad \circ\circ\circ\circ\circ\cdots \\ 1 +_o \omega :&\quad \bullet\circ\circ\circ\circ\cdots \\ \omega +_o 1 :&\quad \circ\circ\circ\circ\circ\cdots\bullet \\ S(\omega) :&\quad \circ\circ\circ\circ\circ\cdots\circ \end{aligned}$$

It's pretty obvious that as *arrangements*, *i.e.* as linear orders, $1 +_o \omega$ is the same as ω, but $\omega +_o 1$ is the same as $S(\omega)$, which is enough for most people.

For those who insist on notation, here goes:

Let $\blacktriangleleft$ be the linear order on $(\{0\} \times 1)) \cup (\{1\} \times \alpha) = \{(0, 0))\cup\{\, (1, n) \mid n \in \omega\,\}$ given by $(u, x) \blacktriangleleft (w, y)$ if either $u < w$ or $u = w$ and $x < y$. Note that this puts the ordered pair $(0, 0)$ before all the ordered pairs $(1, n)$. Define a function $f : \omega \to \{(0, 0))\cup\{\, (1, n) \mid n \in \omega\,\}$ by $f(0) = (0, 0)$ and $f(n + 1) = (1, n)$ for $n \geq 0$. f is clearly 1–1 and onto, and it's not hard to check that $m < n$ implies that $f(m) \blacktriangleleft f(n)$. (At least if one remembers that we should handle the cases where $m = 0$ and $m \neq 0$ separately.) Thus the order type of $((\{0\} \times 1)) \cup (\{1\} \times \alpha), \blacktriangleleft)$ is ω, so $1 +_o \omega = \omega$.

Let $\triangleleft$ be the linear order on $(\{0\} \times \omega)) \cup (\{1\} \times 1) = \{\,(0,n) \mid n \in \omega\,\} \cup \{(0,0)\}$ by $(u,x) \triangleleft (w,y)$ if either $u < w$ or $u = w$ and $x < y$. Note that this puts the ordered pair $(1,0)$ after all the ordered pairs $(0,n)$. Define a function $g : S(\omega) \to \{\,(0,n) \mid n \in \omega\,\} \cup \{(0,0)\}$ by $g(n) = (0,n)$ for all $n \in \omega$ and $g(\omega) = (1,0)$. g is clearly 1–1 and onto, and it's not hard to check that $m < \beta$ implies that $g(m) \triangleleft g(\beta)$. (This time, as long as one remembers to handle the cases where $\beta = \omega$ and $\beta < \omega$ separately.) Thus the order type of $((\{0\} \times \omega)) \cup (\{1\} \times 1), \triangleleft)$ is $S(\omega)$, so $\omega +_o 1 = S(\omega)$.

Thus $1 +_o \omega = \omega < S(\omega) = \omega +_o 1$. □

2. Show, in detail, that $2 \cdot_o \omega = \omega$ but $\omega \cdot_o 2 = \omega +_o \omega$. *[6]*

Solution. We claim that $2 \cdot_o \omega = \omega$ and that $\omega \cdot_o 2 = \omega +_o \omega$.

Following the hint and using the idea that $\alpha \cdot_o \beta$ replaces each element of β with a copy of all of α, as well as the idea that $\alpha +_o \beta$ simply puts all of (a copy of) α before all of (a copy of) β, we can present the linear orders involved as follows:

$$
\begin{aligned}
2 :&\quad \bullet\,\bullet \\
\omega :&\quad \circ\circ\circ\circ\circ\circ\cdots \\
2 \cdot_o \omega :&\quad \bullet\,\bullet|\bullet\,\bullet|\bullet\,\bullet|\cdots \\
\omega \cdot_o 2 :&\quad \circ\circ\circ\circ\circ\cdots\circ\circ\circ\circ\circ\cdots \\
\omega +_o \omega :&\quad \circ\circ\circ\circ\circ\cdots\circ\circ\circ\circ\circ\cdots
\end{aligned}
$$

It is pretty obvious that as arrangements, ω is just like $2 \cdot_o \omega$ and $\omega \cdot_o 2$ is just like $\omega +_o \omega$.

To accommodate those whose proofs must have notation, here goes:

Let $\prec$ be the linear order on on $2 \times \omega = \{\,(a,n) \mid a \in \{0,1\} \wedge n \in \omega\,\}$ defined by $(a,n) \prec (b,m)$ if either $n < m$ or $n = m$ and $a < b$. Define a function $f : \omega \to 2 \times \omega$ by $f(2n) = (0,n)$ and $f(2n+1) = (1,n)$. It is not hard to check that f is 1–1, onto, and order-preserving [all left to the reader]. It follows that the order type of $(2 \times \omega, \prec)$ is ω, and hence that $2 \cdot_o \omega = \omega$.

To show that $\omega \cdot_o 2 = \omega +_o \omega$, first let α be the ordinal $\{0,1,2,3,\dots,\omega,\omega+1,\omega+2,\omega+3,\dots\}$, where $\omega+1$ is shorthand for $S(\omega)$, $\omega+2$ is shorthand for $S(S(\omega))$, and so on.

Let $\ll$ be the linear order on $\omega \times 2 = \{\,(n,a) \mid n \in \omega \wedge a \in \{0,1\}\,\}$ defined by $(n,a) \ll (m,b)$ if either $a < b$ or $a = b$ and $n < m$. Define a function $f : \alpha \to \omega \times 2$ by $f(n) = (n,0)$ and $f(\omega+n) = (n,1)$. Again, it's not hard to check that f is 1-1, onto, and order-preserving [all left to the reader again]. It follows that the order type of $(\omega \times 2, \ll)$ is α, and hence that $\omega \cdot_o 2 = \alpha$.

Now let $\triangleleft$ be the linear order on $(\{0\} \times \omega) \cup (\{1\} \times \omega) = \{\,(a,n) \mid n \in \omega \wedge a \in \{0,1\}\,\}$ defined by $(a,n) \triangleleft (b,m)$ if either $a < b$ or $a = b$ and $n < m$. Define a function $g : \alpha \to (\{0\} \times \omega) \cup (\{1\} \times \omega)$ by $g(n) = (0,n)$ and $g(\omega+n) = (1,n)$. Once again, it is not hard to check that g is 1-1, onto, and order-preserving [all left to the reader again]. It follows that the order type of $((\{0\} \times \omega) \cup (\{1\} \times \omega), \triangleleft)$ is α, and hence that $\omega +_o \omega = \alpha$.

Thus $\omega \cdot_o 2 = \alpha = \omega +_o \omega$.

The astute will notice that $g \circ f^{-1}$ is a 1–1, onto, and order-preserving function from $\omega \times 2$ to $(\{0\} \times \omega) \cup (\{1\} \times \omega)$, and it is particularly simple to define: $\left(g \circ f^{-1}\right)((n, a)) = (a, n)$. If one noted this, it would follow immediately that the respective linear orders would have to have the same order type, *i.e.* $\omega \cdot_o 2 = \omega +_o \omega$. ■

Hint: Draw pictures of each of the linear orders you're dealing with.