Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #1
Counting the Hard Way

John von Neumann devised the following way of building the natural numbers (*i.e.* the non-negative integers) from pretty much nothing at all. Let $\emptyset$ denote the empty set and let S be the operation on sets defined by $S(x) = x \cup \{x\}$. (That is, $S(x)$ contains every element of x, plus x itself as an element.) Here we go:

$0 = \emptyset$
Given that n has been defined, let $n + 1 = S(n)$.

Let's see what this really means:

$$
\begin{aligned}
0 &= \emptyset \\
1 &= S(0) = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
2 &= S(1) = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\, \emptyset, \{\emptyset\}\,\} \\
3 &= S(2) = 2 \cup \{2\} = \{\, \emptyset, \{\emptyset\}\,\} \cup \{\,\{\, \emptyset, \{\emptyset\}\,\}\,\} = \{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,\} \\
4 &= S(3) = 3 \cup \{3\} = \{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,\} \cup \{\,\{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,\}\,\} \\
&= \{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,, \{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,\}\,\} \\
&\ \ \vdots
\end{aligned}
$$

The price of starting with almost nothing at all – and failing to later adopt some notation (like decimal notation :-) that would make natural numbers more compact and readable – is that one has to deal with some rather cumbersome expressions. For example, imagine writing out

$$\{\, \emptyset, \{\emptyset\}\,\} + \{\, \emptyset, \{\emptyset\}\,\} = \{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,, \{\, \emptyset, \{\emptyset\}, \{\, \emptyset, \{\emptyset\}\,\}\,\}\,\}$$

instead of $2 + 2 = 4$. Von Neumann just wanted a way to define the natural numbers in a minimalist language of set theory in which additional symbols were not available. His definition has some interesting properties, though, a couple of which will be investigated by you for this assignment. In what follows, assume that we use von Neumann's definition to define each natural number.

1. Explain why every $n \geq 0$ has exactly n elements. *[3]*

SOLUTION. By definition, $0 = \emptyset$ has zero elements because it is empty, and $1 = S(0) = 0 \cup \{0\} = \{0\}$ has one element. Note that each is the set consisting of all of its predecessors. ($0 = \emptyset$ because it has no predecessors.) Given that we know that some $n \geq 1$ has n elements and $n = \{0, 1, \dots, n-1\}$, it follows, by definition, that $n+1 = S(n) = n \cup \{n\} = \{0, 1, \dots, n-1\} \cup \{n\} = \{0, 1, \dots, n-1, n\}$ has $n + 1$ elements.

The above bit of reasoning is a slightly informal inductive argument. We'll be seeing a lot of induction at certain points in the course. $\square$

2. How many symbols (counting repetitions) does it take to write out each $n \geq 0$ in purely set-theoretic form? Explain why! *[7]*

NOTE. $0 = \emptyset$ needs only one symbol, namely $\emptyset$; $1 = \{\emptyset\}$ needs three symbols (one $\emptyset$ and the two braces); $2 = \{\, \emptyset,\, \{\emptyset\}\,\}$ needs seven symbols (two $\emptyset$, four braces, and a comma); and so on.

SOLUTION. As observed in the note, 0 needs only one symbol, namely $\emptyset$, and 1 requires three symbols, namely $\{\emptyset\}$. After that, the inductive process $n+1 = S(n) = n \cup \{n\}$ plays out in a uniform way in how it affects the symbols required:

> To write out $n+1$ in purely set-theoretic form, you take the set-theoretic form of $n = \{0, 1, \ldots, n-1\}$, add a comma after the (set-theoretic form of) $n-1$, and then insert a copy of the set-theoretic form of n, just after the new comma to get the purely set-theoretic form of $n+1 = S(n) = n \cup \{n\} = \{0, 1, \ldots, n-1, n\}$. This means that if n required k symbols, $n+1$ requires k symbols for the original n, 1 new comma, and k more symbols for the inserted copy of n, for a total of $2k+1$ symbols.

It follows that writing out $n \geq 0$ in purely set-theoretic form requires $2^{n+1} - 1$ symbols. $n = 0$ requires $2^{0+1} - 1 = 2 - 1 = 1$ symbol and $n = 1$ requires $2^{1+1} - 1 = 4 - 1 = 3$ symbols. After that, given that we know that $n \geq 1$ requires $2^{n+1} - 1$ symbols, $n+1$ will require $2\left(2^{n+1} - 1\right) + 1 = 2^{(n+1)+1} - 2 + 1 = 2^{(n+1)+1} - 1$, as claimed.

Again the reasoning just above is a slightly informal inductive argument. □

NOTE. If your analysis and reasoning looked different from those given above, you are not necessarily (or even likely) to be wrong: in most mathematics problems there is more than one correct way to get the job done. For example, a slightly different way of looking at problem **2** above (you get to guess what it is if you didn't use it) would yield a count of $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for the number of symbols to write n in purely set-theoretic form. Same final answer, but a different way to get there.