Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Take-home Final Examination
Due just before midnight on Friday, 18 December.

Instructions: Do both of parts **0** and **n**, and, if you wish, part $\boldsymbol{\omega}$ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.* If submission via Blackboard fails, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

Part 0. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. a. Give the truth tables for all sixteen possible binary logical connectives. (You should already know $\rightarrow$, $\vee$, $\wedge$, and $\leftrightarrow$. You may assign whatever symbols you wish to the other twelve, just keep them all distinct.) *[5]*

b. For each binary logical connective $\circ$, find a formula equivalent to $A \circ B$ using only the connectives $\vee$ and/or $\neg$, the atomic formulas A and/or B, and parentheses. *[5]*

2. Suppose p is a prime number. Show that $\sqrt{p}$ is irrational. *[10]*

3. Suppose the schnitt A represents the negative real number r. Use A to define the schnitt representing the real number $\dfrac{1}{r}$. *[10]*

4. Suppose $\lhd$ is a (strict) linear order on a set A such that $\|A\| = \|\mathbb{N}\| = \aleph_0$. Use an inductive definition to show that there is a function $f : A \to \mathbb{Q}$ such that for all $a, b \in A$, if $a \lhd b$, then $f(a) <_{\mathbb{Q}} f(b)$. *[10]*

Part n. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Suppose $P(x)$ is a one-place relation in a first-order language. Write a formula in the language that expresses the statement "There are at least four possible values of x for which $P(x)$ is true." *[10]*

6. Let $\mathbb{I} = \{\, a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z} \,\}$. Determine whether $\|\mathbb{I}\| = \|\mathbb{N}\|$ or not. *[10]*

7. Suppose $n > 1$ is a natural number such that $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, and $r = 9 \cdot 2^{2n-1} - 1$ are all prime numbers. Show that $a = p \cdot q \cdot 2^n$ and $b = r \cdot 2^n$ are a pair of *amicable numbers*, that is, each is the sum of the other's divisors (other than the other itself). *[10]*

Sadly, there are more questions on page 2 ...

... and here they are:

8. **a.** Give an example of ordinals α, β, and γ such that $\alpha \neq \beta$ but $\alpha +_o \gamma = \beta +_o \gamma$. *[4]*

b. Show that for any ordinals α, β, and γ, if $\gamma +_o \alpha = \gamma +_o \beta$, then $\alpha = \beta$. *[6]*

9. Consider the two sequences $\{a_n\}$ and $\{b_n\}$ defined by

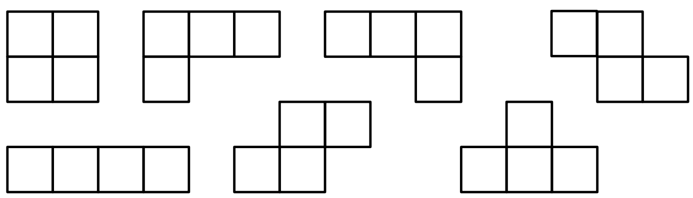
$a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, and by
$b_0 = 0$, $b_1 = 1$, and $b_n = 2a_{n-1}$ for $n \geq 2$, respectively.

Use induction to verify that $a_n \leq b_n$ for all $n \geq 0$. *[10]*

10. A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet eight inhabitants: Peggy, Sue, Carl, Homer, Marge, Betty, Rex and Dave. Peggy says that neither Carl nor Dave are knaves. Sue says, "I know that Betty is a knight and that Rex is a knave." Carl says, "Peggy and Rex are both knights or both knaves." Homer says that at least one of the following is true: that Dave is a knave or that Betty is a knave. Marge claims that Rex and Sue are both knights. Betty claims that it's not the case that Sue is a knave. Rex says that Peggy could claim that Homer is a knave. Dave claims, "Betty and Rex are not the same."

Determine, as best you can, which of the eight are knights and which are knaves. *[10]*

11. *Tetrominoes* are shapes obtained by glueing four 1×1 squares together full edge to full edge. In some cases, such as the game *Tetris*, two tetrominoes that can be made congruent via rotations are considered to be the same, but reflections (*i.e.* flips) are not allowed. This gives seven different tetrominoes:



a. Show how to completely cover an 8×8 square with non-overlapping tetrominoes, using each tetromino at least once and without having any extend beyond the 8×8 square, or explain why no such covering can exist. *[5]*

b. Show how to completely cover a 9×10 rectangle with non-overlapping tetrominoes, using each tetromino at least once and without having any extend beyond the 9×10 rectangle, or explain why no such covering can exist. *[5]*

[Total = 80]

Part $\boldsymbol{\omega}$. Bonus!

$\boldsymbol{\mu}$. Write an original poem about logic or mathematics. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. HAVE A GREAT BREAK!