Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Assignment #$\mathbb{Q}\cap(-\infty, \mathbf{9})$
Cut to the $\mathbb{Q}$uick!
Due on Friday, 20 November.

Recall from various lectures that a *schnitt* or *Dedekind cut* is a subset S of $\mathbb{Q}$ such that:

1. $S \neq \emptyset$ and $S \neq \mathbb{Q}$.
2. S is "closed downwards": if $q \in S$ and $p \in \mathbb{Q}$ with $p < q$, then $p \in S$.
3. S has no largest element: if $q \in S$, then there is an $r \in S$ with $q < r$.

Intuitively, a schnitt S is $(-\infty, s) \cap \mathbb{Q}$ for some real number s. Formally, the real number in question *is* the schnitt S. The set of real numbers is then $\mathbb{R} = \{ S \subset \mathbb{Q} \mid S \text{ is a schnitt} \}$. It is pretty easy to define particular real numbers as schnitts, *e.g.* $0_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q < 0 \}$ and $1_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q < 1 \}$, the operation of addition by $S + T = \{ a + b \mid a \in S \text{ and } b \in T \}$, and the linear order on the reals by $S \leq T \iff S \subseteq T$.

1. Show that if S is a schnitt, then $S +_{\mathbb{R}} (-S) = 0_{\mathbb{R}}$. [Left unfinished in lecture ...] *[4]*

2. Define $\cdot_{\mathbb{R}}$, *i.e.* multiplication on the real numbers as defined via schnitts. [You need not develop the properties of multiplication, just define it fully.] *[6]*

Hint: First, define multiplication between two positive real numbers. Second, use what you did (and a bit of how you want multiplication to behave) to extend the definition to the cases where one or both of the numbers being multiplied is not positive.