Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Assignment #2+2
An almost universe of set theory
Due on Friday, 9 October.

Let's define sets V_n for all $n \geq 0$ as follows:

$V_0 = \emptyset$
If V_n has been defined for some $n \geq 0$, let $V_{n+1} = \mathcal{P}(V_n) = \{ X \mid X \subseteq V_n \}$.

Thus

$$\begin{aligned}
V_0 &= \emptyset \\
V_1 &= \mathcal{P}(V_0) = \mathcal{P}(\emptyset) = \{\emptyset\} \\
V_2 &= \mathcal{P}(V_1) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}(V_2) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

and so on. For anyone who noticed that $V_0 = \emptyset = 0$, $V_1 = S(0) = 1$, and $V_2 = S(1) = 2$, please note that this pattern stops at $n = 3$: $3 = S(2) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subsetneq \mathcal{P}(V_2) = V_3$. It is, however, true that for all $n \geq 0$, we have that 0, 1, 2, ... , n are all elements of V_{n+1}, so it turns out that $n \subseteq V_n$ for all $n \geq 0$.

1. Give an inductive argument showing that $V_n \subseteq V_{n+1}$ for all $n \geq 0$. *[4]*

What happens when we put all of the V_ns together? To put it another way, what is the "limit" of the increasing sequence of sets $V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$? It's the following object:

$$V_\omega = \bigcup_{n=0}^{\infty} V_n = \{ x \mid x \in V_n \text{ for some } n \geq 0 \}$$

V_ω is an infinite set since $\mathbb{N} = \{0, 1, 2, 3, \dots\} \subseteq V_\omega$. (Why?) In fact, it is large and complex enough that most of the axioms of set theory would be true if the universe of sets were V_ω.

2. Which of the axioms of set theory, as developed in the set theory lectures I-IV, would be true if the universe of sets was V_ω? Explain why or why not for each. *[6]*