Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2020
Assignment #$S(S(S(S(S(S(S(S(S(S(0))))))))))$
A Little Ordinal Arithmetic
Due on Friday, 27 November.

Recall from lecture that if $\triangleleft$ is a well-order on a set A, then the *order type* of $(A, \triangleleft)$ is the unique ordinal α such that there is a 1–1 onto function $f : \alpha \to A$ such that for all $\gamma, \beta \in \alpha$, if $\gamma < \beta$, then $f(\gamma) \triangleleft f(\beta)$. We will use this basic idea to define addition and multiplication for ordinals as follows.

$+_o$: Suppose α and β are any two ordinals. Intuitively, $\alpha +_o \beta$ is the order type of putting a copy of β right after after everything in α. Formally, we define $\blacktriangleleft$ on $(\{0\} \times \alpha)) \cup (\{1\} \times \beta)$ by $(u, x) \blacktriangleleft (w, y)$ if either $u < w$ or $u = w$ and $x < y$. Then $\alpha +_o \beta$ is the order type of $((\{0\} \times \alpha)) \cup (\{1\} \times \beta), \blacktriangleleft)$.

$\cdot_o$: Suppose α and β are any two ordinals. Intuitively, $\alpha \cdot_o \beta$ is the order type of inserting a separate copy of all of α in place of each element of β. Formally, we define $\prec$ on $\alpha \times \beta$ by $(a, b) \prec (c, d)$ if either $b < d$ or $b = d$ and $a < c$. Then $\alpha \cdot_o \beta$ is the order type of $(\alpha \times \beta, \prec)$.

When you stick to finite ordinals, *i.e.* natural numbers, these definitions end up giving the same results as $+_{\mathbb{N}}$ and $\cdot_{\mathbb{N}}$. However, both operations behave differently when applied to infinite ordinals, as you will work out below.

Recall that we denote the first infinite ordinal, $\mathbb{N}$, by ω when we think of it as an ordinal in its own right rather than just as the set of natural numbers.

1. Show, in detail, that $1 +_o \omega = \omega < \omega + 1$. *[4]*

2. Show, in detail, that $2 \cdot_o \omega = \omega$ but $\omega \cdot_o 2 = \omega +_o \omega$. *[6]*

Hint: Draw pictures of each of the linear orders you're dealing with.