Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2020

Assignment #1
Counting the Hard Way
Due on Friday, 18 September.

John von Neumann devised the following way of building the natural numbers (*i.e.* the non-negative integers) from pretty much nothing at all. Let $\emptyset$ denote the empty set and let S be the operation on sets defined by $S(x) = x \cup \{x\}$. (That is, $S(x)$ contains every element of x, plus x itself as an element.) Here we go:

$0 = \emptyset$
Given that n has been defined, let $n+1 = S(n)$.

Let's see what this really means:

$$\begin{aligned}
0 &= \emptyset \\
1 &= S(0) = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
2 &= S(1) = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\,\emptyset,\ \{\emptyset\}\,\} \\
3 &= S(2) = 2 \cup \{2\} = \{\,\emptyset,\ \{\emptyset\}\,\} \cup \{\,\{\,\emptyset,\ \{\emptyset\}\,\}\,\} = \{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,\} \\
4 &= S(3) = 3 \cup \{3\} = \{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,\} \cup \{\,\{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,\}\,\} \\
&= \{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,,\ \{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,\}\,\} \\
&\ \vdots
\end{aligned}$$

The price of starting with almost nothing at all – and failing to later adopt some notation (like decimal notation :-) that would make natural numbers more compact and readable – is that one has to deal with some rather cumbersome expressions. For example, imagine writing out

$$\{\,\emptyset,\ \{\emptyset\}\,\} + \{\,\emptyset,\ \{\emptyset\}\,\} = \{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,,\ \{\,\emptyset,\ \{\emptyset\},\ \{\,\emptyset,\ \{\emptyset\}\,\}\,\}\,\}$$

instead of $2 + 2 = 4$. Von Neumann just wanted a way to define the natural numbers in a minimalist language of set theory in which additional symbols were not available. His definition has some interesting properties, though, a couple of which will be investigated by you for this assignment. In what follows, assume that we use von Neumann's definition to define each natural number.

1. Explain why every $n \geq 0$ has exactly n elements. *[3]*

2. How many symbols (counting repetitions) does it take to write out each $n \geq 0$ in purely set-theoretic form? Explain why! *[7]*

NOTE. $0 = \emptyset$ needs only one symbol, namely $\emptyset$; $1 = \{\emptyset\}$ needs three symbols (one $\emptyset$ and the two braces); $2 = \{\,\emptyset,\ \{\emptyset\}\,\}$ needs seven symbols (two $\emptyset$, four braces, and a comma); and so on.