Cardinals IV

2020-11-29

Recap: 1) Sets A & B have the same cardinality, $\|A\| = \|B\|$, if there is a 1-1 onto function $f: A \to B$.

2) A has cardinality less than or equal to B, $\|A\| \leq \|B\|$, if there is a 1-1 function $f: A \to B$.

3) (Schröder-Bernstein Thm) $\|A\| \leq \|B\| \leq \|A\| \Rightarrow \|A\| = \|B\|$.

4) A cardinal is an ordinal α s.t. $\|\beta\| < \|\alpha\|$ for all $\beta \in \alpha$.

(One consequence of the ordinals being the sets of their predecessors is that α is a cardinal iff $\alpha = \{\beta \mid \beta \text{ is an ordinal } \& \|\beta\| < \|\alpha\|\}$.

$\Rightarrow$ The first "uncountable" ordinal is

$$\omega_1 = \aleph_1 = \{\alpha \mid \|\alpha\| \leq \|\omega\|\}.)$$

Q.: Can we put every set into 1-1 correspondence with some ordinal? If not, then there are sets with cardinalities that don't fit the scale of cardinals...

A: Yes, if we add another axiom to the axioms of set theory. (Possibly no, if not...)

Theorem: The following statements are equivalent:

(1) For every set A, there is some ordinal α s.t. $\|\alpha\| = \|A\|$.
[ie so that there is a 1-1 onto function $f: \alpha \to A$]

(2) [Well-ordering Theorem] For any set A, there is a linear order $\trianglelefteq$ which is a well-order of A.

(3) [Axiom of Choice] For any set A, there is a choice function on A; ie a function $f: A \to \bigcup A$ such that if $x \in A$ and $x \neq \phi$, then $f(x) \in x$. ("f chooses an element of every non-empty set in A")

proof: $\boxed{(1) \Rightarrow (2)}$: Assume (1) and suppose A is a set. By (1), there is a function $f: \alpha \to A$ which is 1-1 & onto. Define $\trianglelefteq$ on A by $a \trianglelefteq b$ iff $f^{-1}(a) < f^{-1}(\beta)$ [where $<$ is the usual well-order on α, ie $<$ is $\in$]. Since $<$ is a well-order, so is $\trianglelefteq$ [Exercise].

$\boxed{(2) \Rightarrow (3)}$: Assume (2) and suppose A is a set.

Define $f: A \to \bigcup A = \{x \mid x \in a \in A\}$

by setting $f(x)$ to be the $\lhd$-least element of $x \neq \emptyset$

for the linear order (which is a well-order) for $\bigcup A$, $\lhd$,

that (2) guarantees exists.

($\lhd$ is a well-order on $\bigcup A$ (by (2)) and each $x \in A$ is a subset of $\bigcup A$, so we can use the least element property on $x \neq \emptyset$.)

This defines a choice function on A.

$\boxed{(3) \Rightarrow (1)}$: Suppose that (3) holds and A is a set.

We'll define a function from A to the ordinals as follows:

Let $\cancel{f: A \to \bigcup A}$ $f: \mathcal{P}(A) \to A$ be a choice function. Define

$g: \cancel{A \to ON}$ $ON \to A$ by (transfinite) induction;

Let $g(0) = f(A)$

Given that $g(\beta)$ has been defined for all $\beta < \gamma$ for some ordinal γ, let $g(\gamma) = f(A \setminus \{g(\beta) \mid \beta < \gamma\})$.

This defines a function $g: ON \to A$, but the domain of g cannot be all of ON [since this is too big to be a set]. The least ordinal α for which $g(\alpha)$ is undefined is the α we want. It works because $g: \alpha \to A$ is 1-1 & onto by its definition. Thus $\|\alpha\| = \|A\|$. //

Usually, the Axiom of Choice (AC), ie (3), is added to the axioms we have so far to give the system ZFC – Zermelo-Fraenkel Axioms with the Axiom of Choice for set theory.

Interesting point: Not everyone accepts ZFC (especially AC), but the alternatives don't seem compelling.

Fact: AC cannot be proven from the other axioms [Paul Cohen, 1963]
or disproven — " —. [Kurt Gödel, 1936]