Cardinals II

Thm: (Georg Cantor, 1874) $\|A\| < \|\mathcal{P}(A)\|$

proof: Since $f(a) = \{a\}$ (for all $a \in A$) maps A 1-1 into $\mathcal{P}(A)$, we have $\|A\| \leq \|\mathcal{P}(A)\|$.

Suppose, by way of contradiction, that there was a 1-1 onto function $g: A \to \mathcal{P}(A)$. Let $X = \{a \in A \mid a \notin g(a)\}$.

Since g is allegedly 1-1 & onto $\mathcal{P}(A)$, there is some $x \in A$ such that $g(x) = X$. Then $x \in X \iff x \notin g(x) = X$, a contradiction

Thus there is no 1-1 onto function between A and $\mathcal{P}(A)$, so $\|A\| \neq \|\mathcal{P}(A)\|$, and so $\|A\| < \|\mathcal{P}(A)\|$. //

So far, we've only talked about "Cardinality", not "Cardinals".

Defn: An ordinal κ is a cardinal if $\|\alpha\| < \|\kappa\|$ for all $\alpha < \kappa$.

Note: So the cardinals, as a subcollection of the ordinals, are well-ordered by the order on the ordinals (ie $\in$).

Notation & terminology:

An ordinal that is finite is a cardinal, i.e. every natural number is a cardinal. (Exercise: prove this!).

The infinite cardinals are usually denoted by

aleph → $\aleph_\alpha$ or ω_α for the α^{th} infinite cardinal.

So $\omega_0 = \aleph_0 = \omega = \mathbb{N}$ ...

Prop: $\omega = \omega_0 = \aleph_0$ is a cardinal.

proof: By definition, ω is a cardinal if for every $n<\omega$, we have $\|n\| < \|\omega\|$. This last is true since each $n<\omega$ is finite and ω is infinite... //

Notation: Generic cardinals are often referred to by Greek letters from the middle of the alphabet,

eg $\kappa, \lambda, \mu, \ldots$

Prop.: Suppose $\alpha \geq \omega$ (ie α is an infinite ordinal).
Then $S(\alpha)$ is not ~~a~~ a cardinal.

proof: We'll do the case that $\alpha = \omega$ first.

$$\omega = \{0, 1, 2, 3, \ldots\} \ \& \ S(\omega) = \omega \cup \{\omega\}$$
$$= \{0, 1, 2, 3, \ldots, \omega\}$$

But we know that $f : S(\omega) \to \omega$

defined by $f(\omega) = 0$

& $f(n) = n+1$ for $n \in \omega$

is 1-1 & obviously onto, so $\|\omega\| = \|S(\omega)\|$, so $\|\omega\| \nless \|S(\omega)\|$.
Hence, by definition, $S(\omega)$ is not a cardinal.

Now suppose $\alpha > \omega$. Again, $S(\alpha) = \alpha \cup \{\alpha\}$. Define a 1-1 onto function $g : S(\alpha) \to \alpha$ by $g(\alpha) = 0$,

$g(n) = n+1$ for all $n \in \omega \subsetneq \alpha$,

& $g(\beta) = \beta$ for all $\beta \in \alpha \setminus \omega$.

$\therefore$ By definition, $\|\alpha\| \nless \|S(\alpha)\|$ since $\|\alpha\| = \|S(\alpha)\|$. //

Note, however, that if n is finite (ie $n<\omega$), then n is a cardinal and so is $S(n)=n+1$.

<u>Thm:</u> If A is a set of cardinals, the $\bigcup A$ is also a cardinal.

<u>proof:</u> <u>Case 1:</u> A has a maximum element, say κ.

Then $\bigcup A=\kappa$ and for any $\underset{\bigcup A}{\overset{\shortparallel}{\alpha<\kappa}}$, $\|\alpha\|<\|\kappa\|$ since κ is a cardinal....

<u>Case 2:</u> A has no maximum element.

Suppose we let $\kappa=\bigcup A$; then κ is an ordinal.

If $\alpha<\kappa$, then $\alpha\in\lambda$ for some $\lambda\in A$.

Then $\|\alpha\|<\|\lambda\|$ since $\lambda\in A$ is a cardinal.

Since $\lambda\subseteq\bigcup A=\kappa$, it follows that

$$\|\alpha\|<\|\lambda\|\leq\|\kappa\|.$$

$\therefore$ $\kappa=\bigcup A$ is a cardinal. //

<u>Next:</u> Schröder-Bernstein Thm: $\|A\|\leq\|B\|\leq\|A\| \Rightarrow \|A\|=\|B\|$.