Ordinals III - Transfinite Induction and Order Types

Transfinite Induction: Suppose that $\varphi(x)$ is a statement such that for all ordinals β [$<$ some ordinal δ]

if $\varphi(\alpha)$ is true for all $\alpha < \beta$, then $\varphi(\beta)$ is true.

Then $\varphi(\gamma)$ will be true for all ordinals γ. [$<\delta$]

proof: This relies on the fact that every non-empty collection of ordinals has a least element. Suppose, by way of contradiction, that $\varphi(\gamma)$ fails for some γ. We can pick the least such γ, call it β. Then, since it is least, for all $\alpha < \beta$, $\varphi(\alpha)$ is true. But then $\varphi(\beta)$ is both true and false, a contradiction. //

Def'n: A well-ordered set $(A, \triangleleft)$ has order type α, where α is an ordinal, if there is a 1-1 onto function $f: \alpha \to A$ such that for all $\beta, \gamma \in \alpha$, $\beta < \gamma \Rightarrow f(\beta) \triangleleft f(\gamma)$.

Theorem: Every well-ordered set has an unique order type.

proof: Suppose $(A, \lhd)$ is a well-order. We will define
$f: ON \to A$ as follows: [the domain will end up being less than all of ON, unless A is a proper class]

$f(0)$ = least element of A (in the well-order $\lhd$)

Suppose $f(\alpha)$ has been defined for all $\alpha < \beta$, [if not, stop!]
and $\{f(\alpha) \mid \alpha < \beta\} \neq A$; [A hasn't been used up]

Then $f(\beta)$ = least element of $A \setminus \{f(\alpha) \mid \alpha < \beta\}$.

1° f is 1-1: If $\delta < \gamma$ and $f(\delta)$ & $f(\gamma)$ are defined, then $f(\delta) \lhd f(\gamma)$...

2° f is onto: [Assumes A is a set.] Suppose some $a \in A$ does not have a $\delta \in ON$ s.t. $f(\delta) = a$.
Let b = minimal element of $\{a \in A \mid a$ is not in the range of $f\}$.

Then for every $c \lhd b$, we have $f(c) \in ON$. Consider $\gamma = C = \{f(c) \mid c \lhd b\}$. This is an ordinal (it's a set of ordinals with no gaps & including 0). Then $f(\gamma) = b$, contradiction.

$\therefore$ $\alpha = \{f(\beta) \mid f(\beta) \in A\}$ is an ordinal,
and $f : \alpha \to A$ which is 1-1 & onto, and hence
α is an order type for $(A, \triangleleft)$.

Claim: There is only one such α.

Suppose not; say we had $f : \alpha \to A$
& $g : \beta \to A$
both of which are 1-1 & onto & order-preserving.

Then each of f & g is invertible, so we can
define $h = g^{-1} \circ f : \alpha \to \beta$ which is also 1-1 & onto.
Since $h(0) = 0$ (f takes 0 to the least element of A, & g^{-1} takes it to the least element of β, ie to 0)

& if $h(\gamma) = \gamma$ for all $\gamma < \delta$, then $h(\gamma) = \gamma$
in the same way,

it follows that $h(\gamma) = \gamma$ for all $\gamma \in \alpha$ & since h is onto, it follows that $\beta = \alpha$. //