Ordinals II

We need to finish the proof that if α & β are ordinals, then $\alpha \subsetneq \beta \iff \alpha \in \beta$.

$\boxed{\Leftarrow}$ Finished last time.

$\boxed{\Rightarrow}$ Suppose $\alpha \subsetneq \beta$. Then $\beta \setminus \alpha = \{x \in \beta \mid x \notin \alpha\} \neq \emptyset$.
Since $\in$ is a well-order on β, $\beta \setminus \alpha$ has a least element b.

Claim: $b = \alpha$ [which we'll show by showing $b \subseteq \alpha$ & $\alpha \subseteq b$]

1° $b \subseteq \alpha$: Suppose $x \in b$. Then $x \notin \beta \setminus \alpha$, since otherwise x would be smaller than b and be in $\beta \setminus \alpha$, but b is the least element of $\beta \setminus \alpha$. Since $x \in b$ & $b \in \beta$, $x \in \beta$ by transitivity, so $x \in \alpha$. Thus $b \subseteq \alpha$.

2° $\alpha \subseteq b$: Suppose $x \in \alpha$. Then $x \in \beta$ since $\alpha \subsetneq \beta$, but $x \notin \beta \setminus \alpha$. Since $b \in \beta$ too, trichotomy tells us that exactly one of $x \in b$, $x = b$, or $b \in x$ holds.

a) If $\cancel{x\in\alpha}$ $b \in x$, then $b \in x \in \alpha$, so $b \in \alpha$.

Then $b \notin \beta \setminus \alpha$, contradicting the fact that $\beta \setminus \alpha$ has b as its least element.

$\therefore$ $b \notin x$.

b) If $b = x$, then $b = x \in \alpha$. Then, as above, $b \notin \beta \setminus \alpha$, contradicting the choice of b.

$\therefore$ $b \neq x$.

Thus it must be the case that $x \in b$.

$\therefore$ $\alpha \subseteq b$.

Hence the least element of $\beta \setminus \alpha$ is $\alpha = b$, so in particular, $\alpha \in \beta \setminus \alpha$, ie $\alpha \in \beta$, as desired. //

Notation: The collection of all ordinals is usually called ON. [We can't use $\mathbb{N}$, since that's taken, as is $\mathbb{O}$, which is usually used for the octernions/octonions, which are the number system after the quaternions.]

It turns out that ON is too large to be a set, as we'll see.

Prop: ~~Suppose~~ $\in$ is a well-order of ON.

proof: We first show that $\in$ is a linear order on ON.

1) $\in$ is irreflexive: If $\alpha \in ON$, $\alpha \notin \alpha$ by Foundation.

2) $\in$ is transitive: If $\alpha \in \beta$ & $\beta \in \gamma$ (for $\alpha, \beta, \gamma \in ON$), then $\alpha \in \gamma$ because γ is an ordinal.

3) $\in$ satisfies trichotomy: Suppose that $\alpha, \beta \in ON$. We need to check that exactly one of $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$, holds.

3) $\in$ is trichotomous on ON: Suppose $\alpha, \beta \in ON$.

We need to check that exactly one of $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$ is true.

If $\alpha = \beta$, then $\alpha \notin \beta$ and $\beta \notin \alpha$ by the Axiom of Foundation.

If $\alpha \neq \beta$, then at least one of $\alpha \setminus \beta$ or $\beta \setminus \alpha$ is not empty.

i) If $\alpha \setminus \beta \neq \emptyset$. Then $\alpha \setminus \beta$ has a least element a.

Observe that $a \subseteq \beta$ because

$x \in a \Rightarrow x \notin \alpha \setminus \beta$ [since a is the least element of $\alpha \setminus \beta$]

$\Rightarrow x \in a \in \alpha \setminus \beta \subseteq \alpha$, so $x \in \alpha$, so we must have $x \in \beta$

$\therefore a \subseteq \beta$.

Since $a \notin \beta$, we know that we can't have $a \subsetneq \beta$ by a previous proposition.

(since $a \subsetneq \beta \leftrightarrow a \in \beta$ since a & β are ordinals)

$\therefore$ $\cancel{\alpha} = \beta$, i.e. $\beta \in \alpha$.

ii) If $\beta \setminus \alpha \neq \emptyset$, a similar argument shows that
$\alpha \in \beta$.

So if $\alpha \neq \beta$, then one of $\alpha \in \beta$ or $\beta \in \alpha$ must be true. They can't both be true (ie $\alpha \in \beta \in \alpha \ldots$) by one of the consequences of the Axiom of Foundation.

$\therefore$ If $\alpha, \beta \in ON$, then exactly one of $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$ is true, ie trichotomy holds.

Hence $\in$ is a linear order on ON. To check that it is a well-order we need to show that every non-empty subcollection of ON has a least element.

Suppose A is a non-empty subcollection of ON.
Pick an $\alpha \in A$ and consider $\alpha \cap A$.

1) If $\alpha \cap A = \emptyset$, then α is the $\in$-least element of A, so we've done.

2) If $\alpha \cap A \neq \emptyset$, then by the fact that α is an ordinal, $\alpha \cap A$ has an $\in$-least element γ. But then $\gamma \in A$ and γ must be less that every other element of A (since any lesser element of A would also have to be in $\alpha \cap A$, contradicting γ be the least such.).

Either way, we have a least element!

$\therefore$ $\in$ is a well-order of ON. //

<u>Prop:</u> Suppose A is a non-empty set of ordinals. Then $\bigcup A = \bigcup_{\alpha \in A} \alpha$ is an ordinal, and is the least upper bound of A in ON.

proof: Let $\gamma = \bigcup A$. We claim that γ is an ordinal.

(2) γ is downward closed under $\in$:
Suppose $x \in y \in \gamma$. Since $\gamma = \bigcup A$, there is some $\alpha \in A$ s.t. $y \in \alpha$. Since α is an ordinal and $x \in y \in \alpha$, $x \in \alpha$. But then $x \in \alpha \subseteq \bigcup A = \gamma$.

(1) γ is well-ordered by $\in$: Since γ is the union of a non-empty set of ordinals, $\gamma \subsetneqq ON$, which are well-ordered by $\in$. Since $\in$ linearly orders ON, it linearly orders every subcollection of ON, and γ in particular. γ has the least element property since if $B \subseteq \gamma$ & $B \neq \emptyset$, we have $B \subsetneqq ON$ & has a least element when $B \neq \emptyset$, because $\in$ is a well-order of ON.

$\therefore$ γ is an ordinal. //

Corollary: ON is not a set.

proof: If ON was a set, then $\bigcup ON$ would be an ordinal and then we'd have $\bigcup ON \in ON$.
Also, we know that $S(\bigcup ON) = (\bigcup ON) \cup \{\bigcup ON\}$ would be an ordinal

$$\bigcup ON \in S(\bigcup ON) \in ON$$
$$\subseteq \bigcup ON$$

$\therefore$ $\bigcup ON \in \bigcup ON$ $\otimes$ to Foundation. //

Why ordinals? One reason is to have a framework for induction that goes past infinity.