Ordinals, or, the natural numbers extended "to infinity, and beyond!"

[With apologies to the Toy Story franchise...]

Def'n: A linear order $<$ on a set A is a <u>well-order</u> ([illegible]) if every non-empty subset of A has a least element.

Lemma: $<_{\mathbb{N}}$ is a well-order on $\mathbb{N}$.

proof: Suppose that $X \subseteq \mathbb{N}$ and $X \neq \emptyset$.
By the Foundation Axiom, there is some $a \in X$ s.t. $a \cap X = \emptyset$. (Claim: a is the $<_{\mathbb{N}}$-least element of X.)
Suppose $b \in X$ and $a \neq b$. The linear order $<_{\mathbb{N}}$ is really $\in$, by definition. Since $a \cap X = \emptyset$, $b \notin a$. ie $b \not<_{\mathbb{N}} a$ Since $<_{\mathbb{N}}$ is a linear order, it satisfies trichotomy, and since $a \neq b$ & $b \not<_{\mathbb{N}} a$, we must have $a <_{\mathbb{N}} b$. Thus a is the least element of X. //

<u>Def'n</u>: An <u>ordinal</u> is a set α such that

(1) $\in$ is a well-order on α

& (2) α is closed downward under $\in$ ie $a \in b$ & $b \in \alpha$ $\Rightarrow a \in \alpha$.

Notation: We usually use lower-case Greek letters to represent ordinals.

Our lemma tells us that $\mathbb{N}$ satisfy (1) of being an ordinal. Since we already know that, if $k \in n$ & $n \in \mathbb{N}$, that $k \in \mathbb{N}$ too (Since $n = \{0, 1, \dots, n-1\} \subseteq \mathbb{N} \dots$), $\mathbb{N}$ also satisfies (2), so $\mathbb{N}$ is an ordinal.

<u>Prop</u>: If α is an ordinal and $b \in \alpha$, then b is an ordinal too.

<u>proof</u>: Suppose α is an ordinal and $b \in \alpha$. Since $a \in b$ & $b \in \alpha \Rightarrow a \in \alpha$, we have $b \subsetneq \alpha$, so it is linearly ordered by $\in$ because α is. b is also well-ordered by $\in$ since every nonempty subset of b is also a non-empty subset of α. Thus $\in$ satisfies (1) on b.

We claim that $\in$ abs satisfies (2), i.e. b is downward closed under $\in$: Suppose $x \in b$ & $y \in x$.
Since $x \in b$ & $b \in \alpha$, and α is an ordinal, $x \in \alpha$.
Since $y \in x$ & $x \in \alpha$, and α is an ordinal, $y \in \alpha$.
Then $y \in x$ & $x \in b$ for elements $y, x, b \in \alpha$,
so by transitivity ~~trichotomy~~ (as $\in$ is a linear order on α),
$y \in b$. Thus b and $\in$ satisfy (2) as well.
$\therefore$ $b \in \alpha$ is an ordinal if α is. //

<u>Corollary:</u> If $n \in \mathbb{N}$, then n is an ordinal.
<u>proof:</u> $\mathbb{N}$ is an ordinal... //

<u>Prop:</u> If α is an ordinal, then $S(\alpha) = \alpha \cup \{\alpha\}$ is an ordinal.
<u>proof:</u> (1) $\in$ is a well-order on $S(\alpha)$: First, check it's a linear order.
1) $\in$ is irreflexive: If $a \in S(\alpha)$, then $a \notin a$ by the Axiom of Foundation.

2) $\in$ is transitive: Suppose $a \in b$ and $b \in c$ for $a, b, c \in S(\alpha)$.

case i) $a, b, c \in \alpha$: Then $a \in c$ because $\in$ is a linear order on α.

case ii) $c = \alpha$: then $a \in b$ & $b \in \alpha$, so $a \in \alpha$ by the fact that α is an ordinal, so α is downward closed under $\in$.

$\therefore$ $\in$ is transitive on $S(\alpha)$

3) $\in$ satisfies trichotomy: Suppose $a, b \in S(\alpha)$.

case i) $a, b \in \alpha$: then exactly one of $a \in b$, $a = b$, or $b \in a$ holds because $\in$ is a linear on α.

case ii) At least one of a, b is α. Suppose $b = \alpha$.

Then a) if $a = \alpha$ too, then $a = b$ and neither $a \in b$ nor $b \in a$ happens by Foundation

b) $a \neq \alpha$, in which case $a \in \alpha$ and neither $b \in a$ (by Foundation) nor $b = a$ (by hypothesis) occurs.

$\therefore$ $\in$ satisfies trichotomy on $S(\alpha)$. //

Prop: If α and β are ordinals, then

$$\alpha \subsetneq \beta \iff \alpha \in \beta.$$

proof: $\boxed{\Leftarrow}$ Suppose $\alpha \in \beta$. Then $\alpha \neq \beta$ by Foundation.

$\alpha \subset \beta$ because if $a \in \alpha \in \beta$, then $a \in \beta$

because β is an ordinal (by (2) of the def'n, downward closure under $\in$).

Thus $\alpha \subsetneq \beta$.

$\boxed{\Rightarrow}$ Suppose $\alpha \subsetneq \beta$. Then $\beta \setminus \alpha = \{x \in \beta \mid x \notin \alpha\} \neq \emptyset$.

Since $\in$ is a well-order on β, $\beta \setminus \alpha$ has a least element b.

Then if $x \in b$, (so $x \in \beta$) then $x \in \alpha$ since otherwise x would be a smaller element of $\beta \setminus \alpha$, and so $b \subseteq \alpha$.

On the other hand, suppose $x \in \alpha$. Then $x \in \beta$ too.

By trichotomy, $x \in b$, or $x = b$, or $b \in x$. If $x = b$,

then $b \in \alpha$,

To be finished next time

due to unintelligence moments

of the instructors.

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