Real Numbers III - Arithmetic

Def'n: Suppose S & T are schnitts. Then

$$S +_{\mathbb{R}} T = \{ s+t \mid s \in S \ \& \ t \in T \}.$$

Lemma: If S & T are schnitts, then so is $S +_{\mathbb{R}} T$.

proof: Suppose S & T are schnitts. We need to check that $S +_{\mathbb{R}} T$ satisfies the conditions for being a schnitt.

0) Since S & $T \subseteq \mathbb{Q}$, $\{s+t \mid s \in S \ \& \ t \in T\} \subseteq \mathbb{Q}$.
If $a \notin S$ & $b \notin T$, then $a+b \notin \{s+t \mid s \in S \& t \in T\}$.
(Why? Because $s < a$ & $t < b$ for all $s \in S$ & $t \in T$, so $s+t < a+b$, so $a+b$ is greater than all the elements of $S +_{\mathbb{R}} T$ ...)

1) Suppose that $s+t \in S +_{\mathbb{R}} T$ and $q < s+t$ for some $q \in \mathbb{Q}$. Then $q = \underbrace{s}_{S} + \underbrace{(q-s)}_{T}$, where $q - s < t$
$\therefore$ $q \in S +_{\mathbb{R}} T$. So $S +_{\mathbb{R}} T$ is closed downwards.

2) $S +_{\mathbb{R}} T$ has no largest element. ②

If $s+t \in S +_{\mathbb{R}} T$, where $s \in S$ & $t \in T$, then there are $u \in S$ & $v \in T$ s.t. $s < u$ & $t < v$. But then $s+t < u+v \in S +_{\mathbb{R}} T$. Thus $S +_{\mathbb{R}} T$ has no largest element.

$\therefore$ $S +_{\mathbb{R}} T$ <u>is</u> a schnitt. //

<u>Prop.</u>: $+_{\mathbb{R}}$ is commutative and associative.

<u>proof</u>: 1) $+_{\mathbb{R}}$ is commutative: If S & T are schnitts, then

$$S +_{\mathbb{R}} T = \{s+t \mid s \in S \,\&\, t \in T\} = \{t+s \mid t \in T \,\&\, s \in S\} = T +_{\mathbb{R}} S. \checkmark$$

2) $+_{\mathbb{R}}$ is associative: If S, T, U are schnitts, then

$$(S +_{\mathbb{R}} T) +_{\mathbb{R}} U = \{s+t \mid s \in S \,\&\, t \in T\} +_{\mathbb{R}} U = \{(s+t)+u \mid s \in S \,\&\, t \in T \,\&\, u \in U\}$$

$$= \{s+(t+u) \mid s \in S \,\&\, t \in T \,\&\, u \in U\} = S +_{\mathbb{R}} \{t+u \mid t \in T \,\&\, u \in U\}$$

$$= S +_{\mathbb{R}} (T +_{\mathbb{R}} U). \checkmark$$

//

Def'n: $0_{\mathbb{R}} = \{q \in \mathbb{Q} \mid q < 0\}$ [Exercise: show this a schnitt.]

Lemma: If S is schnitt, then $S +_{\mathbb{R}} 0_{\mathbb{R}} = S$.

proof: We'll show $S +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq S$ and that $S \subseteq S +_{\mathbb{R}} 0_{\mathbb{R}}$.

1) Suppose $t \in S +_{\mathbb{R}} 0_{\mathbb{R}}$. Then $t = s + q$ for some $s \in S$ and $q < 0$. But then $t < s$, so $t \in S$ by the downward closure of S. Thus $S +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq S$.

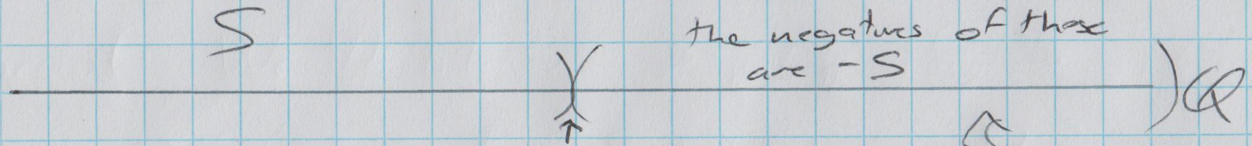
2) Suppose $s \in S$. Since S has no largest element, there is some $t \in S$ s.t. $s < t$. Then

$s = t + (s - t)$ but $t \in S$ & $s - t < 0$

so $s \in S +_{\mathbb{R}} 0_{\mathbb{R}}$. Thus $S \subseteq S +_{\mathbb{R}} 0_{\mathbb{R}}$.

$\therefore$ $S +_{\mathbb{R}} 0_{\mathbb{R}} = S$. //

Q.: How do we define $-S$ for a schnitt S so that $S +_{\mathbb{R}} (-S) = 0_{\mathbb{R}}$?



If S represents a rational, then has a least element...

We get around this by defining $-S$ by cases:

Defn: If S is a schnitt, then

$$-S = \begin{cases} \{-x \mid x \notin S \;\&\; x \neq p\} & \text{if } S = \{q \in \mathbb{Q} \mid q < p\} \text{ for some } p \in \mathbb{Q}. \\ \{-x \mid x \notin S\} & \text{otherwise} \end{cases}$$

Lemma: If S is schnitt, then $-S$ is a schnitt.

proof: Exercise... //

<u>Lemma:</u> If S is a schnitt, then $S +_{\mathbb{R}} (-S) = O_{\mathbb{R}}$.

<u>proof:</u> Suppose $s \in S$ and $x \in -S$.

Then $-x \notin S$ by the definition of $-S$.

$\therefore$ (By the downward closure of S) $s < -x$

$$s + x < (-x) + x = x + (-x) = 0$$

$\therefore$ so $s + x \in O_{\mathbb{R}}$. Thus $S +_{\mathbb{R}} (-S) \subseteq O_{\mathbb{R}}$.

Suppose $g \in O_{\mathbb{R}}$, ie $g < 0$.

Then we need to write g as an element of S plus an element of $-S$.

Then $s + g < s$, so $s + g \in S$ by downward closure.

$g = (s+g) + (-s)$. Is $-s \in -S$? ?

Exercise! Finish this or find another way. //