Real Numbers II - the linear order ("Get in line!")

Recap: A schnitt is a set S such that

(0) $\emptyset \neq S \subsetneq \mathbb{Q}$,

(1) S is closed downwards, ie $p < q \in S \Rightarrow p \in S$,

& (2) S has no largest element, ie $q \in S \Rightarrow \exists r \in S\,(q < r)$.

[Idea: $\alpha \in \mathbb{R}$ corresponds to $\mathbb{Q} \cap (-\infty, \alpha)$.]

Then we defined $\mathbb{R} = \{ S \in \mathcal{P}(\mathbb{Q}) \mid S \text{ is a schnitt} \}$ and $<_{\mathbb{R}}$ by $S <_{\mathbb{R}} T \iff S \subsetneq T$, and showed that $<_{\mathbb{R}}$ is a linear order.

Proposition: $<_{\mathbb{R}}$ is dense. (ie Given $S <_{\mathbb{R}} T$, there is a schnitt U s.t. $S <_{\mathbb{R}} U$ and $U <_{\mathbb{R}} T$.)

proof: Suppose that S and T are schnitts and $S <_{\mathbb{R}} T$. By definition, this means that $S \subsetneq T$.

Choose some $t \in T$ s.t. $t \notin S$. Observe that since S is downward closed we must have $q < t$ for every $q \in S$.

Since T has no largest element, there is some $u \in T$ with $t < u$.

Define a schnitt $U = \{q \in \mathbb{Q} \mid q < u\}$.

Then $S \subsetneq U$ since $t \in U$ but $t \notin S$.

Also, since T has no lagest element, there is some $w \in T$ with $u < w$, so $U \subsetneq T$.

Why is U a schnitt?

(0) $U \neq \emptyset$ since $t \in U$

& $U \neq \mathbb{Q}$ since $w \notin U$.

(1) U is closed downward: Suppose $q \in U$ & $p < q$.

Then $q \in U \Rightarrow q < u$ & so $p < q \Rightarrow p < u$,

so $p \in U$.

(2) Suppose $q \in U$, ie $q < u$. Then $\frac{q+u}{2} < u$ (so $\frac{q+u}{2} \in U$)

and $q < \frac{q+u}{2}$, so there is an element of U

larger that q.

$\therefore$ U is schnitt and $S <_{\mathbb{R}} U <_{\mathbb{R}} T$.

ie $<_{\mathbb{R}}$ is dense. //

Theorem: Suppose a set $A \neq \emptyset$ of real numbers (ie schnitts) has an upper bound (ie there is a schnitt U such that $S <_{\mathbb{R}} U$ for all $S \in A$). Then A has a least upper bound.

proof: A is a set of schnitts. Define a new schnitt, L, by $L = \bigcup\{S \mid S \in A\} = \bigcup_{S \in A} S = \bigcup A$.

We claim that (1) L is a schnitt,

(2) L is an upper bound for A,

& (3) L is $\leq_{\mathbb{R}}$ every upper bound of A.

(1) 0) $L \neq \emptyset$ since $A \neq \emptyset$ and each $S \in A$ is non-empty, $\emptyset \neq S \subseteq L$. ($L \neq \mathbb{Q}$ since $L \subseteq U \neq \mathbb{Q}$)

1) L is closed downward: Suppose $q \in L$ and $p < q$. By the definition of L, $q \in S$ for some schnitt $S \in A$. Then ~~all~~ $p \in S \subseteq L$ because S is downward closed.

2) L has no largest element: Suppose $q \in L$.
Then $q \in S$ for some schnitt $S \in A$.
Since S has no largest element, there is some $r \in S$
s.t. $q < r$. Since $S \subseteq L = \bigcup_{S \in A} S$, $r \in L$,
so L has no largest element.

It follows that L is a schnitt.

(2) L is an upper bound for A:
Since every $S \in A$ has $S \subseteq \bigcup A = L$, $S \subseteq L$,
and so $S \leq_{\mathbb{R}} L$. ie L is an upper bound for A.

(3) Suppose W is an upper bound for A, ie $S \leq_{\mathbb{R}} W$ for all $S \in A$, ie $S \leq_{\mathbb{R}} W \iff S \subseteq W$ for all $S \in A$.
This means that $L = \bigcup_{S \in A} S \subseteq W$, ie $L \leq_{\mathbb{R}} W$.

Thus L is a least upper bound for A. //

Practice on these exercises:

1) Show that $\mathbb{R}$ has no endpoints in the order $<_{\mathbb{R}}$.

(This may require knowing that $\mathbb{Q}$ has no endpts. in $<_{\mathbb{Q}}$.)

2) Show that every ~~collection~~ of set $B \subseteq \mathbb{R}$ with a lower bound has a greatest lower bound.

[A little different from the Theorem we did, basically because things like

$$\bigcap_{n=1}^{\infty} \left(-\infty, \tfrac{1}{n}\right) = (-\infty, 0]$$

can happen.]