Real Numbers (via schnitts)

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The two major ways to define real numbers are using

1. equivalence classes of ought-to-be convergent sequences of rational numbers,

and 2. Dedekind cuts or <u>schnitts</u> ("cuts").

(There are other ways, <u>eg</u> defining the reals directly in terms of decimals, but most fall into one of the major categories under the hood.)

We'll do schnitts! These make it much easier to handle the order properties of the real numbers, at some cost to defining and working with the arithmetic operations.

The basic idea behind schnitts is to identify each real number with the set of rationals less than it.

<u>ie</u> $\alpha \in \mathbb{R} \iff \{q \in \mathbb{Q} \mid q < \alpha\}$

The basic problem at first is to characterize such sets of rationals without reference to the reals.

Suppose $\alpha \in \mathbb{R}$. Then $\{q \in \mathbb{Q} \mid q < \alpha\}$
$= \mathbb{Q} \cap (-\infty, \alpha)$.

What are the properties of such an intersection?

(0) It is a subset of $\mathbb{Q}$, but is not all of the rationals. (Every rational $> \alpha$ is not in the set.)
(Note that if α is rational, it is also not in the set.)

(1) It is closed downwards, i.e. if $q \in \{q \in \mathbb{Q} \mid q < \alpha\}$ and $p \in \mathbb{Q}$ s.t. $p < q$, the $p < q < \alpha$, so $p < \alpha$, so $p \in \{q \in \mathbb{Q} \mid q < \alpha\}$.

(2) It has no largest element. If $q < \alpha$ ($q \in \mathbb{Q}$), then we can find a rational p (in fact, infinitely many) between q & α., but then $q < p$ and $p \in \{q \in \mathbb{Q} \mid q < \alpha\}$.

Def'n: A *schnitt* (or *Dedekind cut*) is a set S such that:

(0) $\emptyset \neq S \subsetneq \mathbb{Q}$

(1) [closed downwards] ~~$\forall q \in S\, \exists p \in S\, (q < p)$~~

(2) [no largest element]

$$\forall q \in S\, \exists p \in S\, (q < p)$$

$$\forall q \in S\, \forall p \in \mathbb{Q}\, (p < q \to p \in S).$$

Then the set of real numbers is

$$\mathbb{R} = \{ S \subseteq \mathbb{Q} \mid S \text{ is a schnitt} \},$$

ie. the schnitts are the real numbers.

Def'n: The linear order $<_{\mathbb{R}}$ on $\mathbb{R}$ is defined by

$$S <_{\mathbb{R}} T \iff S \subsetneq T.$$

Prop: $<_{\mathbb{R}}$ is a linear order.

proof: We need to check that $<_{\mathbb{R}}$ is irreflexive, transitive, and satisfies trichotomy.

1) $<_{\mathbb{R}}$ is irreflexive: Suppose S is a schnitt. Then $S = S$, so $S \subsetneq S$ is not true, so $S \not<_{\mathbb{R}} S$.

2) $<_{\mathbb{R}}$ is transitive: Suppose S, T, & U are schnitts and $S <_{\mathbb{R}} T$ and $T <_{\mathbb{R}} U$. By definition, this means that $S \subsetneq T \subsetneq U$, so $S \subsetneq U$, ie $S <_{\mathbb{R}} U$.

3) $<_{\mathbb{R}}$ is trichotomous: Given S and T that are schnitts, we have to show that exactly one of the alternatives $S <_{\mathbb{R}} T$, $S = T$, or $T <_{\mathbb{R}} S$ holds.

ie $S \subsetneq T$, $S = T$, or $T \subsetneq S$.

Suppose that $S \neq T$ and $T \nsubseteq S$. We need to show that $S \subsetneq T$. ie that $s \in S \Rightarrow s \in T$.

Assume, by way of contradiction, that there is an $s \in S$ such that $s \notin T$. Since T is a schnitt, it is closed downwards, so s must be greater than every element of T (otherwise s would be in T...).
i.e. for every $t \in T$, $t <_{\mathbb{Q}} s$, but since S is a schnitt, it is also closed downward, so every $t \in T$ would be in S as well, i.e. $T \subseteq S$, which contradicts $S \neq T$ and $T \nsubseteq S$.

$\therefore$ By contradiction, if $S \neq T$ and $T \nsubseteq S$, then $S \subsetneqq T$.

i.e. at least one of the alternatives $S \subsetneqq T$, $S = T$, $T \subsetneqq S$ holds for any two schnitts S and T.

Why does exactly one hold? a) If $S = T$, then $S \subsetneqq T$ & $T \subsetneqq S$ obviously fail.
b) If $S \subsetneqq T$, then $S = T$ and $T \subsetneqq S$ obviously fail.
c) If $T \subsetneqq S$, then $T = S$ and $S \subsetneqq T$ obviously fail, too.

$\therefore$ $<_{\mathbb{R}}$ is a linear order. //