The Rationals III - the linear order

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<u>Recap:</u> We defined an equivalence relation $\approx$ on $\{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$

by $(a,b) \approx (c,d) \iff ad = bc$ $\quad$ $\left[(a,b) \text{ represents } \frac{a}{b}\right]$

and then let $\mathbb{Q} = \{ [(a,b)]_\approx \mid a,b \in \mathbb{Z} \;\&\; b \neq 0 \}$,

with $0_\mathbb{Q} = [(0,1)]_\approx$, $1_\mathbb{Q} = [(1,1)]_\approx$, $[(a,b)]_\approx \cdot_\mathbb{Q} [(c,d)]_\approx = [(ac,bd)]_\approx$,

and $[(a,b)]_\approx +_\mathbb{Q} [(c,d)]_\approx = [(ad+bc, bd)]_\approx$. Usual properties for all!

<u>Lemma:</u> If $q = [(a,b)]_\approx \neq 0_\mathbb{Q}$, then $a \neq 0$, and

$-q = [(-a,b)]_\approx = [(a,-b)]_\approx$.

<u>proof:</u> Exercise. //

<u>Def'n:</u> $[(a,b)]_\approx <_\mathbb{Q} [(c,d)]_\approx$ (where $b, d > 0$)

$\iff ad < bc$

$\left[\frac{a}{b} < \frac{c}{d} \iff ad < bc \text{ (as long as } d,b > 0) \right]$

We need to check that $<_\mathbb{Q}$ is well-defined and is a linear order.

Lemma: $<_{\mathbb{Q}}$ is well-defined.

proof: Suppose $[(a',b')]_{\sim} = [(a,b)]_{\sim} <_{\mathbb{Q}} [(c,d)]_{\sim} = [(c',d')]$. (where $b',b,d',d>0$.)

[To show: $[(a',b')]_{\sim} <_{\mathbb{Q}} [(c',d')]_{\sim}$]

Note that $[(a,b)]_{\sim} = [(a',b')]_{\sim} \iff ab' = ba'$

& $[(c,d)]_{\sim} = [(c',d')]_{\sim} \iff cd' = dc'$.

$$
\begin{aligned}
[(a,b)]_{\sim} <_{\mathbb{Q}} [(c,d)]_{\sim} &\iff ad < bc \\
&\implies add' < bcd' \\
&\implies add' < bdc' \\
&\implies (ad')d < (bc')d \\
&\implies ad' < bc' \\
&\implies ad'b' < bc'b' \\
&\implies ab'd' < bc'b' \\
&\implies ba'd' < bc'b' \\
&\implies a'd' < c'b' = b'c' \\
&\iff [(a',b')]_{\sim} = [(c',d')]_{\sim}.
\end{aligned}
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This use a cancellation law for $<$ in $\mathbb{Z}$. (which we ought to prove.)

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<u>Proposition:</u> $<_{\mathbb{Q}}$ is a linear order.

<u>proof:</u> We have to check that $<_{\mathbb{Q}}$ is 1) irreflexive,
2) transitive,
and 3) satisfies trichotomy.

1) $[(a,b)]_{\approx} <_{\mathbb{Q}} [(a,b)]_{\approx} \iff ab < ba = ab$

but $<$ is irreflexive on $\mathbb{Z}$.

so $ab \not< ab$ and hence

$[(a,b)]_{\approx} \not<_{\mathbb{Q}} [(a,b)]_{\approx}$. $\therefore <_{\mathbb{Q}}$ is irreflexive.

2) Suppose $[(a,b)]_{\approx} <_{\mathbb{Q}} [(c,d)]_{\approx} <_{\mathbb{Q}} [(e,f)]_{\approx}$ $(b,d,f > 0)$

$\Rightarrow$ $ad < bc$ & $cf < de$ [Want: $af < be$]

$\Rightarrow$ $\underset{(af)(dc)}{adcf} < \underset{(be)(dc)}{bcde}$

$\Rightarrow$ $af < be$ using cancellation for $<$ in $\mathbb{Z}$.
(assumes $c \neq 0$)

$\therefore <_{\mathbb{Q}}$ is transitive. [Exercise: Handle the case where $c=0$.]

3) Suppose $[(a,b)]_\approx$ and $[(c,d)]_\approx$ are in $\mathbb{Q}$. [We may assume $b, d > 0$.] (9)

Then show that exactly one of $[(a,b)]_\approx <_\mathbb{Q} [(c,d)]_\approx$,

— " — $=$ — " —,

or — " — $>_\mathbb{Q}$ — " —,

is true,

By trichotomy for $<$ on $\mathbb{Z}$, exactly one of

$ad < bc$, $ad = bc$, or $ad > bc$ is true.

ie $cb < da$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$[(a,b)]_\approx <_\mathbb{Q} [(c,d)]_\approx$ $\quad$ $[(a,b)]_\approx = [(c,d)]_\approx$ $\quad$ $[(c,d)]_\approx <_\mathbb{Q} [(a,b)]_\approx$

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<u>Proposition:</u> $<_\mathbb{Q}$ is dense, <u>ie</u> For all $p, q \in \mathbb{Q}$, there is an $r \in \mathbb{Q}$ s.t. $p <_\mathbb{Q} r$ & $r <_\mathbb{Q} q$.

Why is this so? Notice that neither $<_\mathbb{N}$ or $<_\mathbb{Z}$ is dense...

proof: Suppose $p = [(a,b)]_{\approx}$ and $q = [(c,d)]_{\approx}$ and $p <_{\mathbb{Q}} q$. [We need to find an $r = [(e,f)]_{\approx}$ s.t. $p <_{\mathbb{Q}} r$ & $r <_{\mathbb{Q}} q$.]

$(b, d > 0)$

[In real life, if $\frac{a}{b} < \frac{c}{d}$, then $r = \dfrac{\frac{a}{b} + \frac{c}{d}}{2} = \dfrac{ad+bc}{2bd}$ will do the job.]

Since $p <_{\mathbb{Q}} q$ $\Leftrightarrow ad < bc$

We'll try $r = [(ad+bc, 2bd)]_{\approx}$.

Is $p <_{\mathbb{Q}} r$?

$$a2bd \overset{?}{<} b(ad+bc) = bad + b^2c$$

$$\Leftrightarrow \underbrace{2ad}_{=\,ad+ad} < ad+bc$$

$$\Leftrightarrow ad < bc$$ which is true since $p <_{\mathbb{Q}} q$. ✓

Is $r <_{\mathbb{Q}} q$?

$$(ad+bc)d \overset{?}{<} (2bd)c$$

$$\Leftrightarrow ad+bc < 2bdc = bdc + bdc$$

$$\Leftrightarrow ad < bc$$ which is true since $p <_{\mathbb{Q}} q$. ✓

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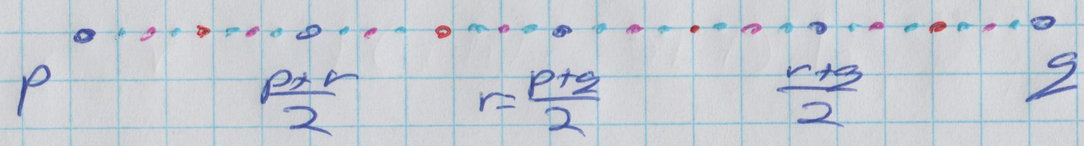
Corollary: Between any two rational numbers there are infinitely many rational numbers

proof: Suppose $p < q$ for $p, q \in \mathbb{Q}$.

Use the Proposition repeatedly:

Informally,



Next time: Define $\mathbb{R}$.