The Integers ($\mathbb{Z}$ is for "Zahlen", ie "integers" in German) 2020-10-16 ①

We'll define the integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, using differences of natural numbers. (Without having subtraction!)

The integer a will be represented by ordered pairs of natural numbers (n, k) such that $n \dot{-} k = a$ ie $n = k + a$. One complication is that different ordered pairs may represent the same integer. (eg $(5,3)$ & $(6,4)$ both represent the integer $2 = 6 \dot{-} 4 = 5 \dot{-} 3$)

We'll use the following as a tool in making this idea work:

<u>Defn:</u> Suppose A is a set and R is a binary relation on A. Then R is an <u>equivalence relation</u> on A if it satisfies:

(1) R is <u>reflexive</u>: $\forall a \in A: \ aRa$

(2) R is <u>transitive</u>: $\forall a,b,c \in A: \ (aRb \wedge bRc) \rightarrow aRc$

(3) R is <u>commutative</u>: $\forall a,b \in A: \ aRb \leftrightarrow bRa$

We'll use an equivalence relation on

$$\mathbb{N}\times\mathbb{N} = \{(a,b) \mid a,b\in\mathbb{N}\}$$

to help define $\mathbb{Z}$: $(a,b)\sim(c,d) \overset{\text{informally}}{\iff} a-b=c-d$

$\overset{\text{formally}}{\iff} a+d=b+c.$

<u>Lemma:</u> $\sim$ is an equivalence relation on $\mathbb{N}\times\mathbb{N}$.

<u>proof:</u> (1) $\sim$ is reflexive: For $(a,b)\in\mathbb{N}\times\mathbb{N}$, $a+b=b+a$ (because $+$ is commutative on $\mathbb{N}$),

so $(a,b)\sim(a,b)$. ✓

(2) $\sim$ is transitive: Suppose $(a,b),(c,d),(e,f)\in\mathbb{N}\times\mathbb{N}$

and $(a,b)\sim(c,d)$ and $(c,d)\sim(e,f)$. [To show: $(a,b)\sim(e,f)$]

$(a,b)\sim(c,d) \iff a+d=b+c$ & $(c,d)\sim(e,f)\iff c+f=d+e$

But then $(a+d)+(c+f) = (b+c)+(d+e)$

$= (a+f)+(d+c) \qquad = (b+e)+(d+c)$

so $a+f=b+e$ by the Cancellation Law for $+$ on $\mathbb{N}$.

Thus, by def'n of $\sim$, $(a,b)\sim(e,f)$. ✓

(3) $\sim$ is commutative: Suppose $(a,b), (c,d) \in \mathbb{N}\times\mathbb{N}$ and

$$(a,b) \sim (c,d).$$

This means that $a+d = b+c \Rightarrow d+a = c+b$ (by commutativity of $+$ on $\mathbb{N}$)

$\Rightarrow c+b = d+a$

$\Rightarrow (c,d) \sim (a,b)$. ✓ //

Def'n: If A is a set and R is an equivalence relation on A, then the <u>equivalence class of $a \in A$</u> is the set $[a]_R = \{ b \in A \mid aRb \}$. (Obviously, since aRa, $a \in [a]_R$.)

Lemma: Suppose A is a set & R is an equivalence relation on A.

Then (1) $aRb \Leftrightarrow [a]_R = [b]_R$

(2) For all a, b; either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \phi$ (but not both!)

proof: (1) $\boxed{\Leftarrow}$ Suppose $[a]_R = [b]_R$. Thus $a, b \in [a]_R = [b]_R$, so, since $b \in [a]_R$, we have aRb.

$\boxed{\Rightarrow}$ If aRb, and $c \in [a]_R$, then aRc,
so bRa & aRc, so bRc, so $c \in [b]_R$.
Similarly, if bRa and $c \in [b]_R$, then aRb & bRc,
so aRc, so $c \in [a]_R$.
Since they have the same elements, aRb (equiv. bRa)
implies that $[a]_R = [b]_R$.

(2) Suppose $a, b \in A$. [To show: that either $[a]_R = [b]_R$
or $[a]_R \cap [b]_R = \emptyset$
(& not both!)]

Suppose $[a]_R \cap [b]_R \neq \emptyset$, ie there is some $c \in [a]_R \cap [b]_R$.
Since $c \in [a]_R$, we have aRc, and since $c \in [b]_R$,
we also have bRc. But then aRc and cRb, so
aRb. By (1), it follows that $[a]_R = [b]_R$. //

We can now define $\mathbb{Z}$ officially:

$$\mathbb{Z} = \{ [(a,b)]_\sim \mid a,b \in \mathbb{N} \}.$$

ie An integer $z \in \mathbb{Z}$ is actually (informally)

$$z = \{ (a,b) \mid a,b \in \mathbb{N} \text{ and } a-b = z \} = [(c,d)]_\sim$$

for any of the ordered pairs (c,d) in z

Next time: We'll define various familiar thing in $\mathbb{Z}$, like 0, 1, $+$, $\cdot$