Natural Numbers IV - More on the linear order

From last time:

We need to show that the (linear-)order on $\mathbb{N}$, given by $n < m$ if and only if $m = n + S(k)$ for some $k \in \mathbb{N}$, satisfies trichotomy: For all $n, m \in \mathbb{N}$, exactly one of $n < m$, $n = m$, or $n > m$ is true.

<u>Proof:</u> We'll first show that at least one of the three must be true. Suppose, by way of contradiction, that for some $n, m \in \mathbb{N}$ all three are false, i.e. $n \not< m$, $n \neq m$, and $m \not< n$. Choose an n for which such an m exists, and let

$$M = \{ m \in \mathbb{N} \mid n \not< m \wedge n \neq m \wedge m \not< n \}.$$ [a set by Comprehension]

We know that $M \neq \emptyset$ by hypothesis. By Foundation, there is an $m \in M$ s.t. $m \cap M = \emptyset$. Consider the following cases:

1° $m = 0$: Then $n = m$ if also $n = 0$, but $n \neq m$, so $n \neq 0$. In this case $n = S(k)$ for some $k \geq 0$. $m = 0 \in n = \{0, \ldots, k\}$ ⊗ to our hypothesis. So $m \neq 0$...

2° $m \neq 0$, then $m = S(k)$ for some $k \geq 0$. (2)

Since $k \in m = S(k) = k \cup \{k\}$, it follows that $k \notin M$.
(as $m \cap M = \emptyset$)

Then at least one of $n < k$, $n = k$, or $n > k$ is true.

a) If $n = k$, then $n = k \in S(k) = m$, so $n < m$ by the Proposition proved the last time; a contradiction to our hypothesis.

b) If $n < k$, then, by def'n, $k = n + S(\ell)$ for some $\ell \in \mathbb{N}$.
Thus $m = S(k) = S(n + S(\ell)) = n + S(S(\ell))$, ie $n < m$, contradicting our hypothesis.

c) If $k < n$, $n = k + S(\ell) = S(k + \ell) = S(\ell + k) = \ell + S(k)$
$= \ell + m = m + \ell$

If $\ell = 0$, then $n = m + \ell = m + 0 = m$, contradicting our hypothesis.

If $\ell \neq 0$, then $\ell = S(a)$ for some $a \in \mathbb{N}$,
so $n = m + \ell = m + S(a)$, ie $m < n$, contradicting our hypothesis.

Since every alternative leads to a contradiction, our assumption that for some m we had all of $n \not< m$, $n \neq m$, and $m \not< n$, is false.

Thus at least one of $n < m$, $n = m$, or $m < n$ is true.

It remains to show that only one of the three alternatives can be true. We'll show that for each alternative, if it's true, the others are false.

1° Suppose $n = m$. We need to show that $n \not< m$ & $m \not< n$.

Assume, by way of contradiction, that $n < m$,

ie $m = n + S(k)$ for some $k \in \mathbb{N}$.

$= S(n+k) = (n+k) \cup \{n+k\}$

thus $n+k \subseteq m$ & $n+k \in m$.

If $k = 0$, then $n + 0 = n \in m$ ⊗ $n = m$ since Foundation implies that no set is an element of itself.

If $k \neq 0$, then $n \in n+k$ [Check by induction]
$\in m$,

so $n \in m$ but Foundation implies that
[(x) to $n=m$] not set is an element of itself.

Thus if $n=m$, then $n \notin m$.

A symmetric argument, reversing the roles of n & m, shows that if $n=m$, then $m \notin n$ either.

Hence if $n=m$, then $n \notin m$ and $m \notin n$.

2° Suppose $n<m$. We need to show that $n \neq m$ and $m \notin n$.

$n<m$ means that $m = n + S(k)$ for some $k \in \mathbb{N}$.
$= S(n+k)$

As before, $n \in S(n+k) = m$, so $n=m$ would contradict Foundation.

$\therefore n \neq m$.

Suppose, on the other hand, that $m<n$,

i.e. $n = m+S(\ell)$ for some $\ell \in \mathbb{N}$.

We already have $m = n+S(k)$ for some $k \in \mathbb{N}$,

so $n = m+S(\ell) = (n+S(k))+S(\ell)$

$= S((n+S(k))+\ell)$

$= n+(S(k)+S(\ell)) = n+S(S(k)+\ell)$

$= S(n+S(S(k)+\ell))$

thus $n \in \underbrace{S(n+S(S(k)+\ell))}_{= n} = (n+S(S(k)+\ell))+\{\ldots\}$

⊗ the Foundation Axiom. (thus if $n<m$, $m \not< n$).

3° Suppose $m<n$. We need to show that $n \neq m$ & $n \not< m$.
A symmetric argument to the one used for 2° above, with the roles of n & m reversed. //

∴ $<$ is a linear order on $\mathbb{N}$. Next time: defining $\mathbb{Z}$.