Natural Numbers III - The linear order

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Today we'll define $<$ on the natural numbers.

$<$ is a binary relation.

<u>Def'n:</u> A <u>k-place relation on a set A</u> is a subset $R \subseteq A^k = \underbrace{A \times A \times \cdots \times A}_{k\text{-times}}$

($A \times A$ is the collection of ordered pairs with both coordinates from A
$\overset{\shortparallel}{A^2}$ and $A^{k+1} = A^k \times A$ for $k \geq 2$.)

In math we mostly use binary (2-place) and unary (1-place) relations. For example, on $\mathbb{N}$ we use $<, |$ (binary) and Even, Odd (unary).

<u>Def'n:</u> $<$ on $\mathbb{N}$ by $a < b$ if and only if $b = a + S(k)$ for some $k \in \mathbb{N}$.

What are the properties of $<$ on $\mathbb{N}$?

<u>Proposition:</u> Suppose $a, b \in \mathbb{N}$ and $a < b$. Then $a \in b$.

<u>proof:</u> If $a < b$, then $a + S(k) = b$, for some $k \in \mathbb{N}$, by def'n of $<$.

Thus $b = a + S(k) = S(a+k)$ by the def'n of $+$.

$= (a+k) \cup \{a+k\}$ [so $a+k \in b$]

but we also know (from before) that

$a+k = \{0, 1, \ldots, a+(k-1)\}$ (if $k > 0$)

$= \{0, 1, \ldots, a-1\}$ (if $k = 0$)

If $k > 0$, $a \in a+k \subseteq b$, so $a \in b$.

If $k = 0$, $a = a+k \in b$. //

<u>Proposition II:</u> Suppose $a, b \in \mathbb{N}$ and $a \in b$. Then $a < b$.

<u>proof:</u> If $a \in b$, then $S(a) = a \cup \{a\} \subseteq b$.

Either $b = S(a) = a + S(0)$ (so $a < b$)

or $b \supsetneq S(a)$, so $b = S(c) = c + S(0)$ for some c with $c \supseteq S(a)$

~~An induction argument can be done~~ $a \in c \Rightarrow a < c < b \Rightarrow a < b$ by transitivity //

Theorem: $<$ is a (strict) linear order on $\mathbb{N}$.

ie $<$ satisfies the following conditions:

1) $<$ is *irreflexive*: for all $n \in \mathbb{N}$, $n \not< n$.

2) $<$ is *transitive*: for all $n, m, k \in \mathbb{N}$,

if $n < m$ and $m < k$, then $n < k$.

3) $<$ is *trichotomous* (ie $<$ satisfies *trichotomy*): for all $n, k \in \mathbb{N}$, exactly one of the three alternatives $n < k$ or $n = k$ or $k < n$ holds.

proof: 1) $<$ is irreflexive: if we had $n < n$ for some $n \in \mathbb{N}$, then we would have that $n \in n$ by the Proposition we just did. But $n \in n$ contradicts one of the first consequences of the Axiom of Foundation. [No set can be an element of itself] $\therefore$ $n \not< n$.

2) $<$ is transitive: Suppose that $n<m$ and $m<k$ for some $n,m,k \in \mathbb{N}$.

By definition this means that $m = n + S(a)$ and $k = m + S(b)$ for some $a,b \in \mathbb{N}$.

Then
$$\begin{aligned} k = m + S(b) &= (n + S(a)) + S(b) \\ &= n + (S(a) + S(b)) \\ &= n + S(\underbrace{S(a) + b}_{\in \mathbb{N}}) \end{aligned}$$

so $n<k$ by def'n.

3) $<$ satisfies trichotomy: We need to show that if $n,m \in \mathbb{N}$ then exactly one of $n<m$, $n=m$, or $m<n$ is true.

a) Show at least one of $n<m$, $n=m$, or $m<n$ is true.

Suppose $n \not< m$ and $n \neq m$. We need to show that $m<n$, i.e. for some $k \in \mathbb{N}$, $n = m + S(k)$.

Next time!