Natural Numbers – Defining addition

Recall that $\mathbb{N}$ satisfies Peano's Axioms:

(1) $0 \in \mathbb{N}$.

(2) For all $k \in \mathbb{N}$, $S(k) \in \mathbb{N}$.

(3) For all $k \in \mathbb{N}$, if $k \neq 0$, then $k = S(n)$ for some $n \in \mathbb{N}$.

(4) $S(k) \neq 0$ for all $k \in \mathbb{N}$.

(5) For all $k, j \in \mathbb{N}$, $k = j \iff S(k) = S(j)$.

(6) If $X \subseteq \mathbb{N}$ and X satisfies axioms (1) & (2), then $X = \mathbb{N}$.

Definition: Define the two-place function on $\mathbb{N}$ as follows:

1. $n + 0 = n$ (for all $n \in \mathbb{N}$)
2. $n + S(k) = S(n+k)$ (for all $n, k \in \mathbb{N}$)

We will proceed to prove a lot of the basic properties of addition.

Theorem: $+$ is associative: $(m+n)+k = m+(n+k)$ for all $n, m, k \in \mathbb{N}$

proof: We'll proceed by induction on k.

Base step: $(k=0)$ Then $m+(n+0) = m+n$ (by 1. of the def'n)

$= (m+n)+0$ (— " —).

Induction Hypothesis: $(m+n)+k = m+(n+k)$ (for some k)

Inductive Step: Assuming the IH, show that

$$(m+n)+S(k) = m+(n+S(k)).$$

$m+(n+S(k)) = m+S(n+k)$ (by 2. of the def'n of $+$)

$= S(m+(n+k))$ (— " —)

$= S((m+n)+k)$ (by the IH)

$= (m+n)+S(k)$ (b 2. of the def'n of $+$).

$\therefore$ By induction, $+$ is associative on $\mathbb{N}$. //

Lemma: For all $n \in \mathbb{N}$, $S(n) = n+1$.

proof:

$$\begin{aligned} S(n) &= S(n+0) && \text{[by 1. of the defn of +]} \\ &= n + S(0) && \text{[by 2. of the defn of +]} \\ &= n+1. && \text{[as } 1 = S(0)\text{]} \end{aligned}$$

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Lemma: For all $n \in \mathbb{N}$, $n+0 = 0+n$.

proof: By induction on n,

Base Step: $(n=0)$

$$\begin{aligned} 0+0 &= 0 && \text{[by 1. of defn of +]} \\ &= 0+0 && \text{[— " —]} \end{aligned}$$

Inductive Hypothesis: $n+0 = 0+n$ for some n

Inductive Step: Show, using the IH, that $S(n)+0 = 0+S(n)$.

$$\begin{aligned} S(n)+0 &= S(n) && \text{[by 1. of the defn of +]} \\ &= S(n+0) && \text{[— " —]} \\ &= \cancel{n+S(0)}\; S(0+n) && \text{[by IH]} \\ &= 0+S(n) && \text{[by 2. of the defn of +]} \end{aligned}$$

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$\therefore$ By induction, $n+0 = 0+n$ for all $n \in \mathbb{N}$.

Thm: For all $n, k \in \mathbb{N}$, $n+k = k+n$. [+ is commutative]

proof: By induction on k:

Base Step: $(k=0)$ $n+0 = 0+n$ (for all n) by the Lemma we just proved...

I.H.: Assume that $n+k = k+n$ (for all n) and some k

Inductive Step: Assuming the IH, show that $S(k)+n = n+S(k)$.

$n+S(k) = S(n+k)$ [by 2. of the def'n of +]

$= S(k+n)$ [by IH]

$= k+S(n)$ [by 2. of the def'n of +]

$= S(n)+k$ [by IH] [works for all n...]

$= S(n+0)+k$ [by 1. of the def'n]

$= (n+S(0))+k$ [by 2. — " —]

$= n+(S(0)+k)$ [by associativity]

$= n+(k+S(0))$ [by IH]

$= n+S(k+0)$ [by 2. of the def'n of +]

$= n+S(k)$ [by 1. of — " —].

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$\therefore$ By induction, $n+k = k+n$ for all $n, k \in \mathbb{N}$.