Set Theory IV - The Natural Numbers

Recall some things from Assignment #1:

1° $S(x) = x \cup \{x\}$ "successor operation"

2° We defined the individual natural numbers as

$$0 = \emptyset$$
$$1 = S(0) = \{0\}$$
$$2 = S(1) = \{0,1\}$$
$$3 = S(2) = \{0,1,2\}$$

Informally, $n+1 = S(n) = \{0,1,2,\ldots,n\}$ (for $n \geq 0$).

Informally, $\mathbb{N} = \{0,1,2,3,\ldots\}$.

How do we actually define the set of natural numbers, or at least characterize it?

Why is there no k which has $k \in X$ but $k \notin \mathbb{N}$
or $k \in \mathbb{N}$ but $k \notin X$?

Suppose $k \in X \setminus \mathbb{N} = \{x \in X \mid x \notin \mathbb{N}\}$

By the Axiom of Foundation there is some $m \in X \setminus \mathbb{N}$
such that $m \cap (X \setminus \mathbb{N}) = \emptyset$.

Note that $0, 1, \ldots, m-1 \in X$ and $\in \mathbb{N}$,
since $\{0, 1, \ldots, m-1\} = m$
$= S(m-1)$ [Note that (3) guarantees the existence of an immediate predecessor "$m-1$".]

Since $m-1 \in X$ & $m-1 \in \mathbb{N}$,
we must have $S(m-1) = m \in X$ and $\in \mathbb{N}$, by (2).
Thus $m \notin \mathbb{N}$ and yet $m \in \mathbb{N}$, a contradiction.
Hence there can be no $k \in X \setminus \mathbb{N}$, i.e. $X \subseteq \mathbb{N}$.
A symmetric argument shows that $\mathbb{N} \subseteq X$ & hence $X = \mathbb{N}$. //

Def'n: $\mathbb{N}$ is the set satisfying the following conditions:

(1) $\phi = 0 \in \mathbb{N}$.

(2) If $k \in \mathbb{N}$, then $S(k) \in \mathbb{N}$.

(3) If $k \in \mathbb{N}$ and $k \neq 0 = \phi$,
then $k = S(n)$ for some $n \in \mathbb{N}$.

Lemma: There can be at most one set satisfying conditions (1)-(3).

proof: Suppose X also satisfies (1)-(3).
By the Axiom of Extension $X = \mathbb{N}$ exactly if

$$x \in X \Leftrightarrow x \in \mathbb{N}.$$

Obviously, by (1): $0 \in X$ and $0 \in \mathbb{N}$.
By (2), if we have $k \in X$ and $k \in \mathbb{N}$,
then $S(k) \in X$ and $S(k) \in \mathbb{N}$.
(So $1 \in X$ & $1 \in \mathbb{N}$, $2 \in X$ & $2 \in \mathbb{N}$, and so on.)

Q.: Why *is* there such a set?

A: We need a new axiom!

Axiom of Infinity: $\mathbb{N}$ exists,

ie there is a set satisfying (1)–(3).

Theorem: $\mathbb{N}$ satisfies Peano's Axioms

(1) $0 \in \mathbb{N}$

(2) For all $k \in \mathbb{N}$, $S(k) \in \mathbb{N}$.

[Not one of Peano's.] (3) If $k \in \mathbb{N}$ and $k \neq 0$, then $k = S(n)$ for some $n \in \mathbb{N}$.

(4) $S(k) \neq 0$ for all $k \in \mathbb{N}$.

(5) For all $k, j \in \mathbb{N}$, $S(j) = S(k) \iff j = k$.

(6) If $X \subseteq \mathbb{N}$ and X satisfies (1) & (2), then $X = \mathbb{N}$.

The major consequence of Peano Axiom (6) is that induction works:

<u>Def'n:</u> An argument by <u>induction</u> has the following basic form: Suppose a statement involving $n \in \mathbb{N}$ as a parameter is $\wp(n)$

Then, if (1) [Base step] $\wp(0)$ is true

and (2) If $\wp(n)$ is true [Inductive Hypothesis] then $\wp(S(n))$ is true (for all $n \geq 0$), [Inductive Step]

then $\wp(n)$ will be true for all $n \in \mathbb{N}$.

<u>Theorem:</u> ~~The~~ Arguments by induction work!

<u>proof:</u> Suppose $\wp(n)$ appears in an argument by induction. Let $X = \{n \in \mathbb{N} \mid \wp(n) \text{ is true}\}$. Then $0 \in X$ because $\wp(0)$, and if $n \in X$, (i.e. $\wp(n)$ is true), then $S(n) \in X$ (since $\wp(S(n))$ must then be true), so by (6) $X = \mathbb{N}$. //