Set Theory III - Functions and the Replacement Axiom

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Recall: (a,b) is defined as $\{\{a\},\{a,b\}\}$

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

"A cross B"

Why is $A \times B$ a set? (If A & B are...)

If A & B are sets, then

0° $A \cup B$ is a set [$A \cup B = \bigcup\{A,B\}$ is a set by the Pair & Union Axioms]

1° $\{a\}, \{a,b\}$ are sets by the Pair Axiom (where $a \in A$ & $b \in B$)

Note that $\{a\}, \{a,b\} \in \mathcal{P}(A \cup B)$.

2° $\{\{a\},\{a,b\}\} = (a,b)$ is a set by the Pair Axiom

Note that $\mathcal{P}(\mathcal{P}(A \cup B)) \ni (a,b)$.

3° $A \times B = \{ z \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a \in A \, \exists b \in B : z = (a,b) \}$

so $A \times B$ is a set by the Comprehension Axiom

<u>Def'n:</u> A <u>function</u> f is a collection (= a set) of ordered pairs such that if $(a,b) \in f$ and $(a,c) \in f$, then $b = c$.

Notice that this definition does not require one to specify a domain or a range up front.

- If f is a function, then the <u>domain</u> of f is the set $D = \{ a \mid \exists b \text{ so that } (a,b) \in f \}$.
- If f is a function, then the <u>range</u> of f is the set $R = \{ b \mid \exists a \text{ so that } (a,b) \in f \}$.

<u>Q:</u> Why are D and R sets if f is a set?

Note that $f \subseteq D \times R$ (assuming that D & R <u>are</u> sets).

$$D \times R \subseteq \mathcal{P}(\mathcal{P}(D \cup R))$$

We'll reverse-engineer $D \cup R$ from f as follows:

f is collection of ordered pairs $(a,b) = \{\{a\},\{a,b\}\}$.

$\bigcup f$ is a collection of all the sets of the form $\{a\}$ and $\{a,b\}$ s.t. $(a,b) \in f$

$\bigcup(\bigcup f)$ is a collection of all the sets a, b s.t. $(a,b) \in f$.

$\shortparallel$

$D \cup R$ Then $D = \{a \in \bigcup(\bigcup f) \mid \exists b \in \bigcup(\bigcup f) : (a,b) \in f\}$

& $R = \{b \in \bigcup(\bigcup f) \mid \exists a \in \bigcup(\bigcup f) : (a,b) \in f\}$

are ~~the~~ sets by the Comprehension Axiom.

This us lets us state the <u>Replacement Axiom</u>

Firstcut: If f is a function and A is a set, with A a subset of the domain of f, then

image of A under f $= f''A = \{f(a) \mid a \in A\}$ is a set

$= \{b \in R \mid a \in A \ \& \ (a,b) \in f\}$. This a set by Comprehension, so this isn't really adding any power to our axioms.

second cut:

Suppose we have a formula $\varphi(x,y)$ with two variables x and y occurring free (unconstrained by a quantifier) in $\varphi(x,y)$. ④

If for some set A, we have that $\varphi(a,y)$ is true of exactly one set y for every $a \in A$, [$\varphi(x,y)$ defines a function f on A]

Axiom of Replacement

then $B = f''A = \{ b \mid \varphi(a,b) \text{ is true for some } a \in A \}$ is also a set.

Next time: The Natural Numbers