Set Theory II - More axioms

Our axioms so far are:

0° Empty Set Axiom: The empty set exists.
(ie There is an empty set.)

(From the logical axioms we can prove $\exists x\,(x=x)$,
from which, using the other axioms we can get an empty set.)

1° Union Axiom: If x & y are sets, then $x \cup y$ is a set.

2° Pairing Axiom: If x & y are sets, then $\{x, y\}$ is a set.

3° Power Set Axiom: If x is a set, then $\mathcal{P}(x) = \{y \mid y \subseteq x\}$ is a set.

4° Extension Axiom: If x & y have exactly the same elements, then $x = y$.

5° Foundation (or Regularity) Axiom: If x is a set, then
either $x = \emptyset$
or there is a $y \in x$ such that $y \cap x = \emptyset$.

<u>Consequence</u>: For any set x, $x \notin x$.

<u>proof</u>: Suppose, by of way contradiction, that $\underline{x \in x}$ for some set x.

Note that $\{x\} = \{x, x\}$ is a set by the Pairing Axiom

$\uparrow$ Extension Axiom.

Direct proof of $x \notin x$

By the Foundation Axiom, $\{x\}$ contains an element with no elements in common with $\{x\}$.

Since $\{x\}$ has x as its only element, ~~The~~ the element given by the Foundation Axiom must be x.

i.e. $x \cap \{x\} = \phi$ i.e. $x \notin x$.

But $x \in x$ by hypothesis, so we have a contradiction.

Thus $x \notin x$.

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6° <u>Comprension Axiom</u> ("Defined subset axiom") ③

Suppose y is a set and $\varphi(x)$ is some condition that may or may not be true of a set x.

Then $\{x \in y \mid \varphi(x) \text{ is true}\}$ is a set.

<u>Q.</u>: Why can we only define subsets of existing sets?

<u>A.</u>: To avoid problems like Russell's Paradox:

Suppose you could define a set by specifying a condition on it's elements without restrictions

Then $y = \{x \mid x \notin x\}$ would be a set.

Then $y \in y \iff y \notin y$, ⊗.

Consequences: If x & y are sets, then $x \cap y$ & $x \cup y$ are sets.

proof: a) $x \cap y = \{ z \in x \mid z \in y \}$ so it's a subset of of x that is definable, so it's a set.

b) If x & y are sets, $\{x, y\}$ is a set (by Pairing Axiom)

and the $\bigcup \{x, y\} = \{ z \mid z \in x \text{ or } x \in y \}$ is a set by the Union Axiom. $\overset{''}{x} \cup y$. //

Proposition: Suppose $x \in y$ (where x & y are, of course, sets.) Then $y \notin x$.

proof: $z = \{x, y\}$ is a set by the Pairing Axiom.

By the Axiom of Foundation, there is an element of z such that z and this element have empty intersection.

$y \cap z = \{x\}$ since $x \in z$ and $x \in y$. So this element can't be y.

So it must be x i.e. $x \cap z = \emptyset$ ~~iz~~ but since $y \in z$ ~~...~~, this means $y \notin x$. //

<u>Define</u> the <u>ordered pair</u> (a,b) (where a,b are sets)

by: ~~$\{a,b\}$ ... but we can't distinguish the order of the elements~~

~~$\{a, \{b\}\}$ ... but we can't distinguish the order if we're unlucky enough~~ to have $a = \{c\}$.

$\{\{a\}, \{a,b\}\}$ works

<u>Exercise:</u> $\{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$ exactly when $a=c$ and $b=d$.

From this we can eventually define relations and functions.