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Basic Set Theory, (semi-) formal and practical

Language(s) for set theory formal (minimalist) and informal (practical)

Variables: $x_0, x_1, x_2, x_3, \ldots$ — $x, y, z, a, b, c, \ldots$

Constants: — $\emptyset, 0, 1, 2, \ldots$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ etc
(We can build most mathematical structures within set theory & so we use their common names.)

Functions: — $f, g, h, \ldots$ $\sin, \cos, e^x, \ldots$ $+, -, /, \circ$ $\cup, \cap, \mathcal{P}, \setminus$

Relations: $=, \in$ — $<, \leq, >, \geq, |, \ldots$ $\subset, \subseteq, \ldots$

Connectives: $\neg, \rightarrow$ — $\wedge, \vee, \Leftrightarrow, \ldots$

Quantifiers: $\forall$ — $\exists$, $\exists!$ ("$\exists! x\, \varphi$" is "there is an unique x s.t. φ")

Grouping symbols: $($, $)$ — $[,], \{, \}$

We also want axioms for set theory that tell us what we can do with sets. Expressible (sometimes horribly) in the formal language, but usually we go informal there too.

(2)

0° **Empty Set Axiom**

Our first axiom is a trivial one, but we have it for convenience.

Informal version: There is an empty set. (ϕ lives!)

Formal version: $\exists y (\forall x (\neg x \in y))$ (semi-formal)

$(\neg(\forall y (\neg(\forall x (\neg x \in y)))))$

1° **Pairing Axiom**

"set-builder notation"

Informally: If a and b are sets, then so is $\{a, b\}$.

Semi-formal: $\forall x \forall y \exists z (x \in z \wedge y \in z \wedge \forall w (w \in z \rightarrow (x = w \vee y = w)))$

Formal: Write it out yourself if you want to...

2° **Union Axiom**

Informally: If A is a set, then so is $\bigcup A = \{x \mid x \in a \text{ for some } a \in A\}$

$= \bigcup_{a \in A} a$.

Semi-formal: $\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x))$ $(y = \bigcup x)$

Applications: We can define the successor operation:

$$S(x) = x \cup \{x\} = \bigcup \{x, \{x\}\} \quad \left(\begin{array}{c}\text{using Union}\\ \text{\& Pairing}\end{array}\right)$$

$$= \bigcup \{x, \{x,x\}\} \quad \left(\begin{array}{c}\{x\} = \{x,x\}\\ \text{by pairing}\end{array}\right)$$

Why is $\{x, x\} = \{x\}$? Because two sets are equal if the same elements..

$x \in \{x\}$
$x \in \{x\}$ and $x \in \{x,x\}$

There is another axiom to capture this idea.

3° Extension Axiom ("Two sets are equal if..." Axiom)

Informally: If a & b are sets, then $a = b$ exactly when they have the same elements.

Semiformally: $\forall x \forall y ((\forall z (z \in x \leftrightarrow z \in y)) \rightarrow x = y)$

4° Power Set Axiom

Informally: If x is a set, then $\mathcal{P}(X) = \{a \mid a \subseteq x\}$

$$= \{a \mid \forall y (y \in a \rightarrow y \in x)\}$$

is a set.

Semi-formally: $\forall x \exists y (\forall z (z \in y \rightarrow \forall w (w \in z \rightarrow w \in x))$

$$\wedge \forall z ((\forall w (w \in z \rightarrow w \in x)) \rightarrow z \in y))$$

Next time: Axiom of Regularity (a.k.a. Axiom of Foundation)