First-Order Logic II : Axioms and deductions

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The <u>logical axioms</u> of a first-order language(s) ~~is~~ are:

(A1) $(\alpha \to (\beta \to \alpha))$ where α, β are formulas of the language

(A2) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$ α, β, γ are formulas

(A3) $(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$ α, β are formulas

(A4) $((\forall x \alpha) \to \alpha^x_t)$

α^x_t is the formula where every occurrence of x in α that is <u>not</u> controlled by a quantifier in α is replaced by a copy of t.

α is a formula
t is a term
x is a variable
t is substitutable for x in α
ie no variable that occurs in t gets captured by a quantifier in α.

(A5) $(\forall x (\alpha \to \beta)) \to ((\forall x \alpha) \to (\forall x \beta))$ α, β formulas

(A6) $(\alpha \to (\forall x \alpha))$ if x does not occur free in α
[ie every occurrence [if any] is controlled by a quantifier in α]

(A7) $(x = x)$ (for any variable x)

(A8) $((x = y) \to (\alpha \to \beta))$

x, y are variable
α, β are atomic formula
and β is obtained from α
by replacing some (or all) (or none)
instances of x in α by y.

Still keep (formally) only one "rule of inference", viz
Modus Ponens (MP):

$$\begin{array}{|ll} (\alpha \to \beta) & \\ \alpha & \\ \hline \beta & \text{MP} \end{array}$$

We'll show using (mostly) the axioms and MP
that if α, then you can deduce $(\exists x\, \alpha)$
(for any variable x).

Fact: Using (A1)–(A3), you can deduce any formula of the form

$$((\lambda \to (\neg \eta)) \to (\eta \to (\neg \lambda)))$$

Exercise: show this.

A variation of contrapositive.
$(((\neg \alpha) \to (\neg \beta)) \to (\beta \to \alpha))$

Assume α.

1. $(((\underbrace{\forall x (\neg \alpha)}_{\lambda}) \to (\underbrace{\neg \alpha}_{\eta})) \to (\alpha \to (\neg (\forall x (\neg \alpha)))))$ — by the fact above

2. $((\forall x (\neg \alpha)) \to (\neg \alpha))$ — (A4) (α is α^x_x, since x is substitutable for itself always)

3. $(\alpha \to (\neg (\forall x (\neg \alpha))))$ — 1,2 MP

4. $(\alpha \to (\exists x \alpha))$ — by the official definition of $(\exists x \alpha)$ as $(\neg (\forall x (\neg \alpha)))$.

5. α — Premiss

6. $(\exists x \alpha)$ — 4,5 M.P.

Even more than with propositional logic, it is not practical to go totally formal in first-order logic.

So in practice, we tend to be pretty informal with first-order arguments.

If α is true, then it ought to be true for any value of x, so in fact it ought to be the case that $\forall x \alpha$ follows, nevermind $\exists x \alpha$.

Fact: (Generalization Theorem) If you can prove α from premisses in which x does not occur free, then you can prove $\forall x \alpha$ from those premisses as well.

Use: You want to show α is true for all x, it suffices to show α is true for some generic x.