First-Order Logic

Propositional logic fails to capture "for all" and "there exists", so we need something better for most of math.

Example: What do we need to talk about number theory?

We need "=", variables "$n, m, k, x, y, \ldots$", constants "0" and "1", functions "S" (successor), $+$, $\cdot$, ... relations "<", "|" (divisibility), ...

We'll still need connectives $\neg, \rightarrow, \wedge, \vee, \leftrightarrow$, and also symbols for the quantifier "for all" ie "$\forall$", and "there exists", ie "$\exists$".

Eg $\quad \forall x\,(x \mid 1)$ [Which isn't true...]

$\forall x\,(1 \mid x)$ [Which is true...]

$\forall x\,((x \mid 1) \rightarrow (x = 1))$ [Which is true...]

Usually, first-order languages are designed to describe particular domains or universes of discourse.

Common elements shared by almost all first-order languages:

1. Variables: $x_0, x_1, x_2, x_3, \ldots$
2. Connectives: $\neg, \rightarrow, \vee, \wedge, \leftrightarrow$
3. Quantifiers: $\forall, \exists$ $\qquad \neg\forall x(\neg \ldots) \equiv \exists x(\ldots)$
4. Grouping symbols: (,)
5. Equality: =

Optional components:

6. constant symbols [name particular things in the universe of discourse]
7. function symbols [name particular functions that have input(s) in the universe of discourse and outputs ← " ⟶]
8. relation symbols: [name relations among the elements of the universe of discourse]

To define the formulas of a first-order language we define the terms of the language.

↳ expression that evaluates out to an element of the universe

Defn: A term of a first order language is defined as follows:

1° If c is a constant, it is a term.

2° If x is a variable, it is a term.

3° If f is a k-place function and $t_1, \ldots, t_k$ are terms, then $f(t_1, t_2, \ldots, t_k)$ is also term.

4° Nothing else is a term. [All terms are built in finitely many steps using 1°–3°.]

Defn: 1° Suppose $t_1, t_2, \ldots, t_k$ are terms and P is a k-place relation, then $P(t_1, t_2, \ldots, t_k)$ is an atomic formula.

2° Suppose t_1, t_2 are terms, then $t_1 = t_2$ is also an atomic formula.

Def'n: A formula of a first order language is given by:

1° Every atomic formula is a formula.

2° If φ is a formula, then $(\neg\varphi)$ is also a formula.

3° If α and β are formulas, then so are $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, and $(\alpha \Leftrightarrow \beta)$, and $(\alpha \to \beta)$.

4° If α is a formula, then $(\forall x\, \varphi)$ and $(\exists x\, \varphi)$ are also formulas.

5° Nothing else is a formula. [ie every formula is built in finitely many steps using 1°–4°.]

e) Language for set theory

It has:

1° variables: $x_0, x_1, x_2, x_3, \ldots$

2° connectives: $\neg, \rightarrow, \wedge, \vee, \leftrightarrow$

3° quantifiers: $\forall, \exists$

4° grouping symbols: $($, $)$ (also $[,], \{, \}, \ldots$)

5° equality: $=$

6° constant symbols: $\emptyset$ ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \omega, \ldots$)

7° function symbols: S, $\cup$, $\cap$, $\mathcal{P}$, $\setminus$

successor, union, intersection, powerset

relative complement $x \setminus y = \{a \in x \mid a \notin y\}$

8° relation symbols: $\in$, $\subseteq$, $\subsetneq$, ...

element of, subset of, proper subset of

Next time: What kind of reasoning can we do in first-order languages?