Propositional Logic II - deductions vs proofs

Left of trying to prove $\alpha \to \alpha$ (from no hypotheses except the axioms)

(‡)

1. $((\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)))$ (A2)
2. $(\alpha \to ((\alpha \to \alpha) \to \alpha))$ (A1)
3. $((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$ 1, 2 MP
4. $(\alpha \to (\alpha \to \alpha))$ (A1)
5. $(\alpha \to \alpha)$ 3, 4 MP

Exercise: Deduce from $\{(\lambda \to (\eta \to \mu)), \eta\}$ that $(\lambda \to \mu)$.

Try this at home and ask about it in seminar or office hours if you need to.

②

(*) Let's deduce $(\alpha \to \gamma)$ from $\{(\alpha \to \beta), (\beta \to \gamma)\}$.

1. $((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma)))$ (A1)

2. $(\beta \to \gamma)$ Premiss (Premise)

3. $(\alpha \to (\beta \to \gamma))$ 1,2 MP

4. $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$ (A2)

5. $((\alpha \to \beta) \to (\alpha \to \gamma))$ 3,4 MP

6. $(\alpha \to \beta)$ Premiss

7. $(\alpha \to \gamma)$ 5,6 MP

Let's prove $((\neg(\neg\beta)) \to \beta)$ from just the logical axioms.

1. $(((\neg\beta) \to (\neg(\neg\beta))) \to (((\neg\beta) \to (\neg\beta)) \to \beta))$ A3

2. $((\neg(\neg\beta)) \to ((\neg\beta) \to (\neg(\neg\beta))))$ A1

3. $((\neg(\neg\beta)) \to (((\neg\beta) \to (\neg\beta)) \to \beta))$ 1,2 using (*)
[we augment the toolkit with previous results]

4. $((\neg\beta) \to (\neg\beta))$ Using (‡)

5. $((\neg(\neg\beta)) \to \beta)$ 3,4 Exercise

<u>Hard</u> exercise: Prove $(\beta \to (\neg(\neg\beta)))$ in this system.
(for possible bonus points if done by yourself)

Obviously, this is not a system we want to be forced to use for any practical purpose: It's too stripped down not to be cumbersome.

In practice, we use anything that can be justified by a truth table & not just Modus Ponens (and the given axiom schema).

eg If you are given $A \wedge B$, you can deduce A (and you can also deduce B)

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Similarly, if you know A (or you know B) then you can deduce $A \vee B$.

Certain common patterns

$(\neg A) \vee (\neg B)$ is equivalent to $\neg(A \wedge B)$

$(\neg A) \wedge (\neg B)$ — " — $\neg(A \vee B)$

$A \vee B$ — " — $(\neg A) \to B$

$A \Leftrightarrow B$ — " — $(A \Rightarrow B) \wedge (B \to A)$

— " — — " — $(A \wedge B) \vee ((\neg A) \wedge (\neg B))$

<u>Contrapositive:</u> $A \to B$ — " — $((\neg B) \to (\neg A))$

A — " — $\neg(\neg A)$

<u>Next time:</u> We will add <u>quantifiers</u>
(variables, constants, relations, functions)