2020-09-16

Propositional Logic, a slightly more formal take on the first half of Chapter 2 of the text.

Propositional logic tries to formalize how connectives (and, or, not, if...then, if and only if) work.

[We'll use "or" inclusively: "or" is true if both parts.]

Atomic formulas (or propositions) are those that can't be broken down any further using connectives.

Symbols: $A_0, A_1, A_2, A_3, \ldots, A_n, \ldots$ [formally]

Informally, use $A, B, C, \ldots$ for these.

We also use grouping symbols: (,)

Informally, also [,]

Connectives

	symbols		other people also use
not	$\neg$	really officially only those	$\sim$
if...then	$\rightarrow$		$\Rightarrow$
and	$\wedge$		&
or	$\vee$		
if and only if	$\leftrightarrow$		$\Leftrightarrow$

We can simulate $\vee$ using $\rightarrow$ and $\neg$: $A \vee B$ is equivalent to $(\neg A) \rightarrow B$.

We can now define the formulas of propositional logic:

1. Every atomic formula is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula. "not φ"
3. If φ and χ are formulas, then $(\varphi \to \chi)$ is a formula. "φ implies χ"

really official

4. If φ and χ are formulas, then so is $(\varphi \wedge \chi)$. "φ and χ"
5. If φ and χ are formulas, then so is $(\varphi \vee \chi)$ "φ or ψ"
6. If φ and χ are formulas, then so is $(\varphi \leftrightarrow \chi)$ "φ if and only if χ"
7. Nothing that isn't defined from atomic formulas in finitely many steps using the rules above counts as a formula.

<u>eg</u> $((A_1 \to A_2) \to (\neg A_1))$ is a formula

$A_1 A_2 \wedge (A_3)$ is not a formula.

If we assign "truth values" ("T or F") to each atomic formula, we can determine the "truth value" of every other formula:

φ	$(\neg\varphi)$
T	F
F	T

α	β	$(\alpha \to \beta)$
T	T	T
T	F	F
F	T	T
F	F	T

α	β	$(\alpha \vee \beta)$
T	T	T
T	F	T
F	T	T
F	F	F

α	β	$(\alpha \wedge \beta)$
T	T	T
T	F	F
F	T	F
F	F	F

α	β	$(\alpha \Leftrightarrow \beta)$
T	T	T
T	F	F
F	T	F
F	F	T

So let's suppose that our formula is

$$((A_1 \to (\neg A_2)) \to (A_3 \to A_1))$$

Then the full truth table requires $2^{\#\text{ different connectives}} = 2^3 = 8$

lines:

A_1	A_2	A_3	$((A_1 \to (\neg A_2)) \to (A_3 \to A_1))$			
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	F	F	F
F	T	F	T	F	T	T
F	F	T	T	T	F	F
F	F	F	T	T	T	T

Formal proofs or <u>deductions</u> in propositional logic.

The logical axioms for <u>really official</u> ($\to$ & $\neg$) propositional logic are:
(logical axiom schema)

(A1) $(\alpha \to (\beta \to \alpha))$ for any formulas α & β

(A2) $(((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$ — " — α, β, γ

(A3) $(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$ — " — α & β

These axioms actually fully capture the truth table definition of $\to$ & $\neg$.

A <u>deduction</u> in proposition logic is a sequence of formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$

such that each φ_k is either (1) a hypothesis of the proposition

or (2) a logical axiom

or (3) follow from preceding φ_i & φ_j using <u>Modus Ponens</u> (MP)

<u>Modus Ponens</u>

From φ and $(\varphi \to \psi)$ one can deduce ψ.

[This is a "rule of procedure" or "deduction rule".]

<u>To show:</u> $(\alpha \to (\alpha \to \alpha))$ can be proved from nothing at all (i.e. just from the logical axioms.

Deduction: 1. $(\alpha \to (\alpha \to \alpha))$ (A1) [with β being α]

This is it!

To show: $(\alpha \to \alpha)$ with no non-logical hypotheses

Try:

1. $(\alpha \to (\alpha \to \alpha))$ (A1)
2. $((\alpha \to (\alpha \to \alpha)) \to ((\alpha \to \alpha) \to (\alpha \to \alpha)))$ (A2)
3. $((\alpha \to \alpha) \to (\alpha \to \alpha))$ 1,2 MP

& now what?

Next time!