Irrational Numbers — a couple of less common arguments

[§1.6 of the textbook — alternate take]

The rational numbers, $\mathbb{Q}$, are the real numbers that can be written as a ratio of integers: $\frac{a}{b}$ (where $b \neq 0$) and $a, b \in \mathbb{Z}$ [today we'll stick to $a, b \in \mathbb{N}$]

A real number is irrational if it is not rational.

How can you tell? Usually we can't eg Is e^{π} rational or not? Unknown.

Sometimes we can, eg $\sqrt{2}$ is irrational.

eg Do an argument by contradiction, ie assume $\sqrt{2}$ is rational and show this leads to absurdity.

So suppose $\sqrt{2} = \frac{a}{b}$, where $a, b > 0$ and integers, & a and b have no factors (other than 1) in common. (If not, cancel them out first.)

$$\sqrt{2} = \frac{a}{b} \Rightarrow a = \sqrt{2} \cdot b \Rightarrow a^2 = 2b^2 \Rightarrow 2 \mid a^2$$

& since 2 is prime, $\Rightarrow 2 \mid a$ ie a is even

a is even $\Rightarrow a = 2k$ for some k.

$\Rightarrow (2k)^2 = a^2 = 2b^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow b^2 = 2k^2 \Rightarrow 2 | b^2$

& since 2 is prime $2|b^2 \Rightarrow 2|b$

$\therefore$ $2|a$ & $2|b$ even though a & b have no
[so 2 is a common factor] common factors other than 1

Since $2 \neq 1$ it follows that we have a contradiction.

Thus the assumption that $\sqrt{2}$ is rational is incorrect, and so $\sqrt{2}$ must be irrational. //

<u>Note</u>: This is not a <u>direct</u> proof.

<u>es</u> $\log_3(2)$ is irrational

<u>proof</u>: Assume not, <u>ie</u> that $\log_3(2) = \frac{a}{b}$ for some positive intger a & b. Then

$$3^{a/b} = 3^{\log_3(2)} = 2 \Rightarrow 3^a = (3^{a/b})^b = 2^b.$$

But 3^a must be odd and 2^b is even, so they can't be equal. Thus $\log_3(2)$ can't be rational. //

⇒ To show: $e = 2.718\ldots$ is irrational

A little bit of calculus: Remainder terms for Taylor polynomials.

Recall that the Taylor series at 0 [MacLaurin series] of $f(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

The k^{th} Taylor polynomial of $f(x)$ at 0 is

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n .$$

Most of the time $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

k^{th} remainder term

$$= T_k(x) + R_k(x)$$

ie $R_k(x) = f(x) - T_k(x)$

Fact: $\displaystyle R_k(x) = \frac{f^{(k+1)}(t)}{(k+1)!} x^{k+1}$ ~~where~~ for some t with $0 < t < x$.

[Langrange form of the remainder]

Recall the the Taylor series of e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

which converges to e^x for all x.

Then $T_k(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} = \sum_{n=0}^{k} \frac{x^n}{n!}$

and $R_k(x) = \frac{f^{(k+1)}(t)}{(k+1)!} x^{k+1} = \frac{e^t}{(k+1)!} x^{k+1}$ for some t with $0 < t < x$.

Thus
$$\begin{aligned} e = e^1 &= T_k(1) + R_k(1) \\ &= 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots + \frac{1^k}{k!} + \frac{e^t}{(k+1)!} 1^{k+1} \quad 0<t<1 \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \frac{e^t}{(k+1)!} \quad \text{—"—} \end{aligned}$$

Note that $e - T_k(1) = R_k(1) = \frac{e^t}{(k+1)!} < \frac{e^1}{(k+1)!} < \frac{3}{(k+1)!}$

Suppose that e was rational ie $e = \frac{a}{b}$ for some positive integers a & b.

Then $\overset{e}{\overset{\shortparallel}{\frac{a}{b}}} - T_k(1) = \frac{a}{b} - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{k!} < \frac{3}{(k+1)!}$

Choose $k+1$ so that $k > 3$ and $k > b$ [eg $k > 3b$]

[Note that $0 < \frac{a}{b} - T_k(1) < \frac{3}{(k+1)!} < 1$.]

Multiply by ~~(k+1)!~~ $k!$:

$$\cancel{\frac{e(k+1)!}{b} - \frac{(k+1)!}{0!} - \frac{(k+1)!}{1!} - \frac{(k+1)!}{2!} - \cdots - \frac{(k+1)!}{k!}} < 3\cdot\frac{(k+1)!}{(k+1)!}$$

$$0 = 0k! < \underbrace{\frac{a\,k!}{b} - \frac{k!}{0!} - \frac{k!}{1!} - \cdots - \frac{k!}{k!}}_{\text{integer}} < \frac{3}{(k+1)!}\cdot k! = \frac{3}{k+1} < 1$$

integer since b is a factor of $k!$

integers

since $k+1 > k > 3$.

Thus $\frac{a}{b} - T_k(1)$ is an integer strictly between 0 & 1.

Not many such beasts, in fact, no such beasts, so the assumption that e was rational must be wrong. //

<u>eg</u> Are there irrational numbers c & d such that c^d is rational?

Yes. <u>eg</u> $e^{\ln(2)} = 2$ (but ought to show that $\ln(2)$ is irrational)

Cheap alternative: (a little indirect)

Consider $\sqrt{2}^{\sqrt{2}}$: $\sqrt{2}$ is irrational, so if $\sqrt{2}^{\sqrt{2}}$ is rational, we're done.

If $\sqrt{2}^{\sqrt{2}}$ is irrational, then

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

does the job.

We're going to look at different proof techniques, starting with Chapter 2 (on logic).

(We'll defer relations (§1.7) until later.)