# Welcome to Math 2200H

## Mathematical Reasoning!

An introduction to the horrors of abstraction and proofs...

The plural of "hoof" is "hooves", so shouldn't the plural of "proof" be "prooves"? English is not consistent...

How to solve problems in math:

4 step Polya's Principles:
1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

Skim through Chapter One of the textbook.

Def'n: $p$ is prime number if $p \geq 2$ and is an integer, such that the only positive integer factors of $p$ are 1 and $p$ $(a|p \Rightarrow a=1 \text{ or } a=p)$

Replacement condition: If $p$ is a factor of $ab$, where $a$ & $b$ are positive integers, then $p$ is a factor of $a$ or $p$ is a factor of $b$ [or both].
$(p|ab \Rightarrow p|a \text{ or } p|b)$

Notation: is a kind of shorthand to keep things compact & readable.

If $n$ & $k$ are integers, then $n|k$ ("$n$ divides $k$") means that $n$ is a factor of $k$, ie $k = n \cdot m$ for some integer $m$.

eg Quadratic formula: The solution(s) of $ax^2+bx+c=0$ (where $a\neq 0$) are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Problem: Suppose $p$ satisfies the "replacement" condition for prime.

Show this!

$$[\, p|ab \Rightarrow p|a \text{ or } p|b \,]$$

Then the only positive integer factors of $p$ are 1 and $p$.

$$[\, n|p \Rightarrow n=p \text{ or } n=1 \,].$$

① Understand the problem?

Here it boils down to understanding the definitions...

[We'll assume that all numbers we deal with are positive integers...]

② Devise a plan? $1^{\underline{c}}$ find a starting point & follow your nose

What does $n|p$ mean? It means $p = n \cdot k$ for some integer $k \geq 0$.

So this means we can try to apply the "replacement" condition since $p$ is a factor of itself because $p = 1 \cdot p$.

③ $\overset{n|p}{\Longrightarrow} p = nk \Rightarrow p | nk \overset{\text{"replacement" condititia}}{\Longrightarrow} p|n$ or $p|k$.

1. If $p|n$, then $n \leq p$ (since $n|p$) and $p \leq n$. This can only happen if $n = p$.

2. If $p|k$, then so $p \leq k$ but $p = nk$ so $k \leq p$ so $k = p$. But if $k = p$, $p = nk = np \Rightarrow n = 1$ (since $p \neq 0$).

$\therefore$ $n|p \Rightarrow n = 1$ or $n = p$.

9 Look back! (Proofread - try to find any mistakes & see if you can improve what you did.)

<u>To show</u>: $n|p \Rightarrow n=p$ or $n=1$ ($p$ is prime) (using the "replacement" def'n)

<u>proof</u>: Note that $a|b \Rightarrow a \leq b$ for positive integers $a$ & $b$.

$n|p \Rightarrow p = nk$ for some positive integer $k$

$\Rightarrow p|n$ or $p|k$ (using the "replacement" condition in the definition of prime)

$\Rightarrow p|n \Rightarrow p \leq n$ and $n|p \Rightarrow n \leq p \Rightarrow p = n$

or $p|k \Rightarrow p \leq k$ and $p=nk \Rightarrow k \leq p \Rightarrow k = p$

$\Rightarrow p = nk = np \Rightarrow n = 1$ (since $p \neq 0$).

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Problem: Suppose $a, b \geq 0$ and are real numbers.

Show that $\sqrt{ab} \leq \frac{a+b}{2}$.

$\uparrow$ geometric mean of a&b

$\uparrow$ arithmetic mean of a&b

1 Understand the problem? Not hard here...

2 Devising a plan?

Try getting rid of the square root...

3 $\sqrt{ab} \leq \frac{a+b}{2}$

$$\iff (\sqrt{ab})^2 = ab \leq \left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4}$$

$$\iff 4ab \leq a^2 + 2ab + b^2$$

$$\iff 0 \leq a^2 - 2ab + b^2 = (a-b)^2$$

4 Look back! Write it so it gets to the desired conclusion from something true.

$$(a-b)^2 \geq 0 \quad \text{since any square} \geq 0$$

expand

$$\Rightarrow a^2 - 2ab + b^2 \geq 0 \overset{\text{add } 4ab \text{ to both sides}}{\Rightarrow} a^2 + 2ab + b^2 \geq 4ab$$

consolidate

$$\Rightarrow (a+b)^2 \geq 4ab \overset{\div 4}{\Rightarrow} \frac{(a+b)^2}{4} \geq ab$$

$$\overset{\sqrt{\ }}{\Rightarrow} \sqrt{\frac{(a+b)^2}{4}} = \frac{a+b}{2} \geq \sqrt{ab} \quad \checkmark \quad //$$

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Book uses the natural numbers as $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the first few chapters, but the switches to the more common $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. We'll use $\mathbb{N} = \{0, 1, 2, \ldots\}$ from the beginning and something like $\mathbb{N}^+ = \mathbb{N}^{\geq 1} = \{1, 2, 3, \ldots\}$.