Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #9
Real Bounds

Recall that a *schnitt* or *Dedekind cut* is a subset S of $\mathbb{Q}$ such that:

1. $S \neq \emptyset$ and $S \neq \mathbb{Q}$.
2. S is "closed downwards": if $q \in S$ and $p \in \mathbb{Q}$ with $p < q$, then $p \in S$.
3. S has no largest element: if $q \in S$, then there is an $r \in S$ with $q < r$.

Intuitively, a schnitt S is $(-\infty, s) \cap \mathbb{Q}$ for some real number s. Formally, the schnitt S *is* the real number s. That is, we define the set of real numbers to be $\mathbb{R} = \{ S \subset \mathbb{Q} \mid S \text{ is a schnitt} \}$. With this definition, it is pretty easy to define certain common real numbers as schnitts, *e.g.* $0_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q < 0 \}$ and $1_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q < 1 \}$, the operation of addition by $S + T = \{ a + b \mid a \in S \text{ and } b \in T \}$, and the linear order on the reals by $S \leq T \iff S \subseteq T$. (Things get more complicated when defining multiplication, unfortunately.)

This assignment is dedicated to showing that the linear order on the real numbers is *complete*: that every non-empty set of real numbers with an upper (respectively, lower) bound has a least upper (respectively, greatest lower) bound. Here are the relevant definitions:

- If $\emptyset \neq A \subset \mathbb{R}$, then $u \in \mathbb{R}$ is an *upper bound* for A if $a \leq u$ for all $a \in A$.
- If $\emptyset \neq A \subset \mathbb{R}$, then $v \in \mathbb{R}$ is a *lower bound* for A if $v \leq a$ for all $a \in A$.
- If $\emptyset \neq A \subset \mathbb{R}$ has an upper bound, then the *supremum* or *least upper bound* of A is the real number $\sup(S)$ such that $\sup(S)$ is an upper bound for A and $\sup(S) \leq u$ for every upper bound u of A.
- If $\emptyset \neq A \subset \mathbb{R}$ has a lower bound, then the *infimum* or *greatest lower bound* of A is the real number $\inf(S)$ such that $\inf(S)$ is a lower bound for A and $v \leq \inf(S)$ for every lower bound v of A.

In what follows, you may assume that the usual algebraic operations and the linear order on the rationals have all the usual properties. (Do note that the linear order on the rationals is *not* complete.)

1. Show that every non-empty set $A \subset \mathbb{R}$ with an upper bound has a supremum. *[5]*

SOLUTION. Suppose $A \subset \mathbb{R}$ and has an upper bound, *i.e.* there is a $w \in \mathbb{R}$ such that $a < w$ for all $a \in A$. The supremum of A, $\sup(A)$, is easy enough to define:

$$\sup(A) = \bigcup A = \bigcup_{a \in A} a = \{ q \in \mathbb{Q} \mid q \in a \text{ for some } a \in A \}$$

That is $\sup(A)$ is just the union of all the schnitts in A. We do need to verify two things: that $\sup(A)$ is a schnitt too, and that it is the least upper bound of A.

We first verify that $\sup(A)$ is a schnitt, condition by condition. Note that by it's definition as a union of schnitts, each of which must be a subset of $\mathbb{Q}$, $\sup(A) \subseteq \mathbb{Q}$.

1. $A \neq \emptyset$, so there is a schnitt $a \in A$. Since a is a schnitt, $a \neq \emptyset$, and since $a \subset \sup(A)$ by the definition of $\sup(A)$, it follows that $\sup(A) \neq \emptyset$.

   Since w is an upper bound for A, we also have that $a \subseteq w$ for each $a \in A$, so it follows that $\sup(A) \subseteq w$. Since w is a schnitt, there is some $p \in \mathbb{Q}$ such that $p \notin w$, and thus $p \notin \sup(A)$ too. It follows that $\sup(A) \neq \mathbb{Q}$.
2. Suppose $q \in \sup(A)$ and $p \in \mathbb{Q}$ with $p < q$. Then $q \in a$ for some schnitt $a \in A$. Since a is closed downwards, $p \in a$, and then $p \in \sup(A)$ because $a \subseteq \sup(A)$ by the definition of $\sup(A)$. Thus $\sup(A)$ is closed downwards.

3. Suppose $q \in \sup(A)$. Then $q \in a$ for some schnitt $a \in A$ and, since a has no largest element, there is a $p \in a$ with $q < p$. We have $p \in \sup(A)$ because $a \subseteq \sup(A)$ by the definition of $\sup(A)$. It follows that $\sup(A)$ has no largest element.

Since it satisfies all three parts of the definition of a schnitt, $\sup(A)$ is indeed a schnitt.

To see that $\sup(A)$ is an upper bound of A, observe that for every $b \in A$, $b \subseteq \bigcup_{a \in A} a = \sup(A)$. To see that $\sup(A)$ is the least upper bound of A, suppose $u \in \mathbb{R}$ is any upper bound of A. This means that $a \subseteq u$ for every $a \in A$, and so $\sup(A) = \bigcup_{a \in A} a \subseteq u$. Thus $\sup(A)$ is less than or equal to any other upper bound for A.

Putting all of this together, $\sup(A)$ as defined above is indeed a schnitt that is the supremum of A. □

2. Show that every non-empty set $A \subset \mathbb{R}$ with a lower bound has an infimum. *[5]*

SOLUTION. Rather than do this directly, we will exploit the result in question **1** by first proving the following lemma:

> LEMMA. A non-empty set $A \subset \mathbb{R}$ has lower bound v if and only if $B = \{ -a \mid a \in A \}$ has upper bound $-v$.
>
> PROOF. For all $a \in A$, $v < a$ if and only $-a < -v$, so if v serves as lower bound for A, v serves as an upper bound for B, and *vice versa.* □
>
> NOTE. We leave it to the interested reader to check that if a and v are schnitts, then $v < a$ if and only if $-a < -v$.

Now suppose that $A \subset \mathbb{R}$ is a non-empty set with a lower bound v. By the lemma, $B = \{ -a \mid a \in A \}$ is a nonempty set of reals which has upper bound $-v$. It follows by **1** that B has a least upper bound $\sup(B)$. Let $\inf(A) = -\sup(B)$. We claim that $\inf(A)$ is the greatest lower bound of A.

First, $\inf(A)$ is a lower bound for B by the lemma because $\inf(A) = -\sup(B)$ and $\sup(B)$ is an upper bound for B.

Second, suppose v is any other lower bound of A. Then, by the lemma, $-v$ is an upper bound of B, so $\sup(B) \leq -v$. It follows that $v = -(-v) \leq -\sup(B) = \inf(A)$. Hence $\inf(A)$ is the greatest lower bound of A.

Thus any non-empty set $A \subset \mathbb{R}$ with a lower bound has an infimum. ■