Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #8
Rational Thoughts

Recall that we defined the rational numbers as follows:

The equivalence relation $\approx$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{ (a,b) \mid a, b \in \mathbb{Z} \wedge b \neq 0 \}$ is defined by $(a,b) \approx (c,d) \iff ad = bc$. Then $\mathbb{Q} = \left\{ [(a,b)]_\approx \mid a, b \in \mathbb{Z} \wedge b \neq 0 \right\}$.

Intuitively, $(a,b) \approx (c,d)$ if the pairs have the same ratio, *i.e.* $\frac{a}{b} = \frac{c}{d}$, so the equivalence class $[(a,b)]_\approx$ represents the ratio $\frac{a}{b}$.

We then defined multiplication on the rational numbers by $[(a,b)]_\approx \cdot [(c,d)]_\approx = [(ac,bd)]_\approx$ and showed that it was commutative. Also, we let $0 = 0_\mathbb{Q} = [(0,1)]_\approx$ and $1 = 1_\mathbb{Q} = [(1,1)]_\approx$.

In what follows, you may assume that the usual algebraic operations on the integers have all the usual properties.

1. Show that non-zero rational numbers have multiplicative inverses: if $p \in \mathbb{Q}$ and $p \neq 0$, then there is a $p^{-1} \in \mathbb{Q}$ such that $pp^{-1} = 1$. *[5]*

SOLUTION. Note first that $0_\mathbb{Q} = [(0,1)]_\approx = \{ (0,b) \mid b \neq 0 \}$ because $(0,1) \approx (a,b) \iff 0 = 0b = 1a = a$. Second, $1_\mathbb{Q} = [(1,1)]_\approx = \{ (a,a) \mid a \neq 0 \}$ because $(1,1) \approx (a,b) \implies a = 1a = 1b = b$.

Now suppose $p \in \mathbb{Q}$ and $p \neq 0$. Then $p = [(a,b)]_\approx$ for some $a, b \in \mathbb{Z}$, with both $a \neq 0$ and $b \neq 0$. Let $p^{-1} = [(b,a)]_\approx$. Since $a \neq 0$, this is a valid equivalence class for $\approx$, and hence $p^{-1} \in \mathbb{Q}$. Finally,

$$pp^{-1} = [(a,b)]_\approx \cdot [(b,a)]_\approx = [(ab,ba)]_\approx = [(ab,ab)]_\approx = 1_\mathbb{Q} = 1\,.$$

Note that we used the commutativity of multiplication in $\mathbb{Z}$. We also implicitly used the fact that if $a, b \in \mathbb{Z}$ and both $a \neq 0$ and $b \neq 0$, then $ab \neq 0$. $\square$

2. Show that multiplication on $\mathbb{Q}$ satisfies the Cancellation Law: if $p, q, r \in \mathbb{Q}$, where $r \neq 0$, and $pr = qr$, then $p = q$. *[5]*

SOLUTION. Suppose $p, q, r \in \mathbb{Q}$, where $r \neq 0$, and $pr = qr$ and assume that we know that multiplication on $\mathbb{Q}$ is associative and that $s1 = s$ for all $s \in \mathbb{Q}$. Since $r \neq 0$, there is an $r^{-1} \in \mathbb{Q}$ such that $rr^{-1} = 1$. Then $p = p1 = p\left(rr^{-1}\right) = (pr)r^{-1} = (qr)r^{-1} = q\left(rr^{-1}\right) = q1 = q$, as desired.

We leave it ti the interested reader to check that multiplication on $\mathbb{Q}$ is associative and that $s1 = s$ for all $s \in \mathbb{Q}$, if that hasn't already been done ... $\blacksquare$