Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2019

Solutions to Assignment #5
The Natural Numbers as a Linear Order

Recall from class that we defined the natural numbers from the empty set $\emptyset$ and the successor function $S(x) = x \cup \{x\}$ as follows: $0 = \emptyset$, $1 = S(0) = 0 \cup \{0\} = \{0\}$, $2 = S(1) = 1 \cup \{1\} = \{0, \{0\}\}$, $3 = S(2) = 2 \cup \{2\} = \{0, \{0\}, \{0, \{0\}\}\}$, and so on. In general, the immediate successor of the natural number n is $S(n) = n \cup \{n\}$. The set of all natural numbers, guaranteed to actually exist by the axiom of infinity, is usually called $\mathbb{N}$.

One of the advantages of this definition of the natural numbers is that is very easy to define the usual linear order $<$, namely $n < k$ if and only if $n \in k$. This assignment is all about showing that this way of defining $<$ works.

1. Show that if n, k, $m \in \mathbb{N}$ and $n \in k$ and $k \in m$, then $n \in m$. *5]*

Solution. We proceed by induction on m.

Base Step. ($m = 0$) The statement "if $n \in k$ and $k \in 0$, then $n \in 0$" is true for all n, $k \in \mathbb{N}$ since the hypothesis of the implication is false because there can be no $k \in \emptyset = 0$.

Inductive Hypothesis. ($m = p$) For all n, $k \in \mathbb{N}$, if $n \in k$ and $k \in p$, then $n \in p$.

Inductive Step. ($m = S(p)$) Assume the Inductive Hypothesis and suppose $n \in k$ and $k \in S(p)$. We need to show that $n \in S(p)$.

Note that $k \in S(p) = p \cup \{p\}$ means that either $k \in p$ or $k \in \{p\}$. In the first case, $n \in p$ by the Inductive HYpothesis, from which it follows that $n \in S(p) = p \cup \{p\}$; in the second case, we have $n \in k = p$, from which it again follows that $n \in S(p) = p \cup \{p\}$. Either way, we have $n \in S(P)$, as required. $\square$

2. Show that if n, $k \in \mathbb{N}$, then exactly one of $n \in k$, $k \in n$, or $n = k$ is true. *[5]*

Note: It may be helpful here or there to use a more sophisticated version of the Axiom of Foundation: *If x is a non-empty set, then there is an element $y \in x$ such that $y \cap x = \emptyset$.* Notice that this disallows having any set b such that $b \in b$: if such a set b existed, then $\{b\}$ would violate the Axiom of Foundation.

Solution. Suppose n, $k \in \mathbb{N}$.

If $n = k$, then the Axiom of Foundation guarantees that $n \notin k$ and $k \notin n$.

If $n \neq k$, we need to show that exactly one of $n \in k$ or $k \in n$ is true. Since $n \neq k$, we must have $n \setminus k \neq \emptyset$ or $k \setminus n \neq \emptyset$ (or both)*. Suppose for the sake of argument, that $n \setminus k \neq \emptyset$. (The other case has a symmetric proof.) By the Axiom of Foundation, as given in the note above, it follows that there is an $m \in n \setminus k$ such that $m \cap (n \setminus k) = \emptyset$. This means that $m \in n$ and $m \notin k$. However if $x \in m$, then $x \in n$ because $m \in n$ by **1** above, and $x \in k$ because otherwise we would have $x \in (n \setminus k)$, contradicting $m \cap (n \setminus k) = \emptyset$. There are two possibilities, by the definition of the natural numbers: $m = 0$ or $m = S(p)$ for some $p \in \mathbb{N}$.

First, if $m = 0$, then $0 \notin k$. It's not too hard to see from the definition of the natural numbers that this is only possible if $k = 0$, but then $k = 0 = m \in n$. Moreover, $n \notin \emptyset = 0 = k$.

Second, suppose $m = S(p)$ for some natural number p. Then $p \in k$ but $m = S(p) \notin k$. Again, it's not too hard to see from the definition of the natural numbers that this is only possible if $k = S(p)$, so we have $k = S(p) = m \in n$. The Axiom of Foundation guarantees that $n \notin k$ in this case: just consider the set $\{n, k\}$ if we had both $n \in k$ and $k \in n$. ■

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* Recall that $A \setminus B = \{\, a \in A \mid a \notin B \,\}$.