Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Solution to Assignment #4
Ordered Pairs

Here is a formal definition of a minimal first-order language for set theory:

The symbols of the language are as follows:

Variables: x_0, x_1, x_2, ...
Connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$
Quantifiers: $\forall$, $\exists$
Parentheses: (,)
Equality: $=$
Set Membership: $\in$ (a 2-place relation)
Just to be paranoid: all of the above symbols are distinct, none is a substring of any other, and there are no other symbols in the language.

The formulas (*i.e.* statements) of the language are defined as follows:

1. For any variables x_i and x_j of the language, $(x_i = x_j)$ and $(x_i \in x_j)$ are formulas of the language.
2. If φ and ψ are any formulas of the language, then $(\neg\varphi)$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are also formulas of the language.
3. If φ is any formula of the language and x_i is any variable of the language, then $(\forall x_i \varphi)$ and $(\exists x_i \varphi)$ are also formulas of the language.
4. No string of symbols of the language is a formula of the language unless it was formed using (possibly many applications of) rules 1–3 above.

This language is inefficient in some ways – it could really use a symbol for the empty set and some additional relations, such as the subset relation, and overuses parentheses, among other things – but as first-order languages go it is pretty uncomplicated. To compensate for its deficiencies, one usually augments this language informally with auxiliary symbols for common objects (*e.g.* $\emptyset$), operations (*e.g.* $\cup$, $\cap$, $\setminus$), and relations (*e.g.* $\subseteq$, $\subsetneq$), as well as *ad hoc* names for generic sets (*e.g.* A, B).

1. Define "ordered pair" in the (formal and unaugmented) given language. *[10]*

NOTE: The ordered pair (a, b) is different from the ordered pair (b, a) unless $a = b$. Your first problem for **1** is to figure out what it actually means to define such a concept in the given language.

SOLUTION. Per the note, we need to define the notion of "ordered pair" in terms of sets, and then write a formula in the given language that is true exactly when a set meets this definition.

First, we define (a, b) to be $\{\{a\}, \{a, b\}\}$.[†] To check that this definition works, we need to show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$. The "if" direction is trivial. For the "only if" direction, suppose that we are given that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. For these two sets to be equal, they must have the same elements. Since each has one element that itself has just one element, this means that we must have $\{a\} = \{c\}$, and hence that $a = c$. Since the two sets are equal, the remaining elements $\{a, b\}$ and $\{c.d\}$ must be equal; since we already know that $a = c$, it follows that we must have $b = d$.

[†] This definition is due to the Polish mathematician Kazimierz Kuratowski, who worked in topology and related areas.

It remains to write a formula that recognizes when a set is of the form $\{\{a\},\{a,b\}\}$. Note that we must be careful to allow for the possibility that $a=b$, in which case $\{\{a\},\{a,b\}\}=\{\{a\}\}$. (Why?) This means, for example, that we can't assume that the set $\{\{a\},\{a,b\}\}$ has exactly two elements. However, it cannot be empty and cannot have more than two elements, it must have one element that is a singleton, *i.e.* a set with exactly one element, and any other element would have to have exactly two elements, one of which is the same as the element of the singleton.

Our final formula will assert that x_0 is $\{\{x_1\},\{x_1,x_2\}\}$ for some x_1 and x_2, leaving open the possibility that $x_1=x_2$. We bulild this formula in pieces, staying totally within the official language (hence the excess of parentheses):

α ["x_0 is not empty"] is $(\exists x_3\,(x_3 \in x_0))$.

β ["x_0 has at most two elements"] is $(\exists x_4\,(\exists x_5\,(\forall x_6\,(((x_6 \in x_0) \to ((x_6 = x_4) \vee (x_6 = x_5)))))))$.

γ ["x_0 has an element that is a singleton"] is:

$$(\exists x_7\,((x_7 \in x_0) \wedge (\exists x_8\,((x_8 \in x_7) \wedge (\forall x_9\,((x_9 \in x_7) \to (x_9 = x_8)))))))$$

δ ["There are sets x_1 and x_2 such that every element of x_0 is $\{x_1\}$ or $\{x_1,x_2\}$"] is:

$$\begin{aligned}
&(\exists x_1\,(\exists x_2\,(\forall x_3\,(((x_3 \in x_0) \to (\\
&((x_1 \in x_3) \wedge (\forall x_4\,((x_4 \in x_3) \to (x_4 = x_1))))\\
&\vee (((x_1 \in x_3) \wedge (x_2 \in x_3)) \wedge (\forall x_5\,((x_5 \in x_3) \to ((x_5 = x_1) \vee (x_5 = x_2)))))\\
&\quad))))))
\end{aligned}$$

Our formula – which has two redundant components [which ones?] – is $(\alpha \wedge (\beta \wedge (\gamma \wedge \delta)))$. ■