Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #3
Connectives

The binary logical connective $\downarrow$ has the following truth table:

A	B	$A \downarrow B$
T	T	F
T	F	F
F	T	F
F	F	T

This connective is variously called the *Peirce arrow* or the *Quine dagger*, or, more descriptively, as *joint denial* or *NOR* ("not or").

1. Show how to write formulas logically equivalent to each of $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$, and $A \leftrightarrow B$ using just the connective $\downarrow$, or explain why it can't be done in each such case. *[5]*

SOLUTION. A formula using $\downarrow$ that is equivalent to $\neg A$ is $A \downarrow A$. We verify this with a simple truth table:

A	$\neg A$	$A \downarrow A$
T	F	F
F	T	T

Since $\neg A$ and $A \downarrow A$ give the same truth values as outputs for all inputs, the two formulas are equivalent.

Similarly, $(A \downarrow B) \downarrow (A \downarrow B)$ is equivalent to $A \vee B$:

A	B	$A \downarrow B$	$(A \downarrow B) \downarrow (A \downarrow B)$	$A \vee B$
T	T	F	T	T
T	F	F	T	T
F	T	F	T	T
F	F	T	F	F

Since $A \wedge B$ is equivalent to $\neg((\neg A) \vee (\neg B))$,

A	B	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$	$\neg((\neg A) \vee (\neg B))$	$A \wedge B$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

and we already know how to write $\neg$ and $\vee$ in terms of $\downarrow$, we can write $A \wedge B$ in terms of $\downarrow$ too. It's pretty ugly, though, as this approach yields the formula:

$$(((A \downarrow A) \downarrow (B \downarrow B)) \downarrow ((A \downarrow A) \downarrow (B \downarrow B))) \downarrow (((A \downarrow A) \downarrow (B \downarrow B)) \downarrow ((A \downarrow A) \downarrow (B \downarrow B)))$$

Feel free to grind out the truth table for this directly if you are feeling mathochistic ... A little fiddling (or looking things up :-) yields a much simpler formula, namely $(A \downarrow A) \downarrow (B \downarrow B)$, which can be readily verified to work using a truth table:

A	B	$A \downarrow A$	$B \downarrow B$	$(A \downarrow A) \downarrow (B \downarrow B)$	$A \wedge B$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	F

Using the above equivalences and the fact that $A \to B$ is equivalent to $(\neg A) \vee B$ (check it for yourself), yields the formula $((A \downarrow A) \downarrow B) \downarrow (((A \downarrow A) \downarrow B))$ equivalent to $A \to B$. You can grind out the truth table, if you feel the need to...

Finally, $A \leftrightarrow B$ is equivalent to $(A \to B) \wedge (B \to A)$, which gives a truly horrendous formula if written out using $\downarrow$ to simulate the connectives $\to$ and $\wedge$ as above. Some fiddling yielded a much more efficient formula using $\downarrow$ to simulate $\leftrightarrow$, namely $((A \downarrow B) \downarrow A) \downarrow ((A \downarrow B) \downarrow B)$. Here's the truth table:

A	B	$A \downarrow B$	$(A \downarrow B) \downarrow A$	$(A \downarrow B) \downarrow B$	$((A \downarrow B) \downarrow A) \downarrow ((A \downarrow B) \downarrow B)$	$A \leftrightarrow B$
T	T	F	F	F	T	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	T	F	F	T	T

□

2. For each of the following pairs of connectives, show how to use that pair of connectives to write a formula logically equivalent to $A \downarrow B$, or explain why it can't be done in each such case: $\{\neg, \vee\}$, $\{\neg, \to\}$, $\{\neg, \wedge\}$, $\{\vee, \wedge\}$, and $\{\vee, \to\}$. *[5]*

SOLUTION. It is possible to use the first three pairs of connectives to simulate $\downarrow$, but it is not possible to use eiether of the last two pairs to do so:

$\{\neg, \vee\}$: As the name NOR for $\downarrow$ suggests, $A \downarrow B$ is equivalent to $\neg(A \vee B)$, which we can readily check with a truth table:

A	B	$A \vee B$	$\neg(A \vee B)$	$A \downarrow B$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

$\{\neg, \to\}$: Since $A \vee B$ is equivalent to $(\neg A) \to B$, what we did above suggests that $\neg((\neg A) \to B)$ should be equivalent to $A \downarrow B$, which we verify by grinding out the truth table:

A	B	$\neg A$	$(\neg A) \to B$	$\neg((\neg A) \to B)$	$A \downarrow B$
T	T	F	T	F	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	F	T	T

$\{\neg, \wedge\}$: From the observation that $A \downarrow B$ is true exactly when both A and B are false, we can guess that $(\neg A) \wedge (\neg B)$ is equivalent to $A \downarrow B$. We verify this by grinding out the truth table:

A	B	$\neg A$	$\neg B$	$(\neg A) \wedge (\neg B)$	$A \downarrow B$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\{\vee, \wedge\}$ and $\{\vee, \to\}$: No combination of $\vee$, $\wedge$, and/or $\to$ can simulate $\downarrow$. The reason is pretty simple: each of these connectives has the property that it returns the truth value T if both inputs are T. It follows that any formula built using just these three connectives will give T when all the atomic formulas in it have truth value T. Since $\downarrow$ returns the truth value F when both inputs are T, it can't be replicated by any formula using only $\vee$, $\wedge$, and/or $\to$. ■

NOTE: In both problems you may use the designated connective(s), as well as the atomic formulas A and B, more than once in each logically equivalent formula.