Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #10
Complex Numbers

Informally, complex numbers are what you get when you add a square root of -1 into the real numbers. That is, they are taken to be numbers of the form $a+bi$, where a, $b \in \mathbb{R}$ and $i^2=-1$. The set of complex numbers is usually denoted by $\mathbb{C}$.

Historically, complex numbers were invented in the 1500s in the course of dealing with the formulas for solving cubic equations by Niccolo Tartaglia and Gerolamo Cardano, in which one had to deal with complex numbers in the course of the computation even if one was only interested in real solutions. Rafael Bombelli, in the later 1500s, was the first to treat the complex numbers as a number system in their own right.

Your task in this assignment is to do the job that history has already done, and get it right this time. :-)

1. Give a precise definition of the complex numbers and the operations of addition and multiplication on the complex numbers. *[6]*

SOLUTION. This is a lot easier than, say, defining the rationals from the integers or the reals from the rationals. No need for equivalence relations or tricky sets ...

First, define $\mathbb{C}=\mathbb{R}\times\mathbb{R}=\{\,(a,b)\mid a,\,b\in\mathbb{R}\,\}$. That is, the informal $a+ib$ is officially just the ordered pair (a,b). In particular, i is the ordered pair $(0,1)$.

Second, define $+$ on $\mathbb{C}$ by $(a,b)+(c,d)=(a+c,b+d)$, *i.e.* $(a+ib)+(c+id)=(a+c)+i(b+d)$. Note that $0_{\mathbb{C}}=(0,0)$.

Third, define $\cdot$ on $\mathbb{C}$ by $(a,b)\cdot(c,d)=(ac-bd,ad+bc)$, *i.e.* $(a+ib)+(c+id)=(ac-bd,ad+bc)$. Note that $1_{\mathbb{C}}=(1,0)$ and $i^2=(0,1)\cdot(0,1)=(0\cdot 0-1\cdot 1,0\cdot 1+1\cdot 0)=(-1,0)=-1_{\mathbb{C}}$. □

2. Suppose $a+bi\in\mathbb{C}$. Find complex numbers equal to $\dfrac{1}{a+bi}$ and $\sqrt{a+bi}$, respectively. *[4]*

SOLUTION. Suppose $a+bi\neq 0$, *i.e.* it is not the case that $a=b=0$. Then $a^2+b^2\neq 0$. Since

$$(a+bi)\cdot\left(\frac{a-bi}{a^2+b^2}\right)=\frac{(a+bi)(a-bi)}{a^2+b^2}=\frac{a^2-abi+bai-b^2i^2}{a^2+b^2}$$
$$=\frac{a^2-b^2(-1)}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}=1\,,$$

it follows that $\dfrac{1}{a+bi}=\dfrac{a-bi}{a^2+b^2}=\dfrac{a}{a^2+b^2}-\dfrac{b}{a^2+b^2}i$. Note that 0 does not have a multiplicative inverse in $\mathbb{C}$.

The square root of $a=a+0i$ is $\pm\sqrt{a}$ if $a\geq 0$ and $\pm\sqrt{|a|}\cdot i$ if $a<0$. For $a+bi$ with $b\neq 0$ the square root is given by $\pm(c+di)$, where

$$c=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}\quad\text{and}\quad d=\operatorname{sign}(b)\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}\,,$$

and $\operatorname{sign}(b)=\begin{cases}+1 & b>0\\ -1 & b<0\end{cases}$. Note that if $b\neq 0$, then $\dfrac{\pm a+\sqrt{a^2+b^2}}{2}$ is guaranteed to be positive. We leave it to the reader to check that this works. ■

NOTE. For both **1** and **2** you may assume that the real numbers and the basic operations on them have been defined and have the usual properties.