Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2019

Solution to Assignment #1
Sums of Squares

First, a bit of terminology: "integer square" is usually taken to mean "the square of an integer". Thus $4 = 2^2$ is an integer square, but 2 is not an integer square because $\sqrt{2}$ is not an integer.

1. Prove that three times the sum of three integer squares can be writtem as the sum of four integer squares. *[10]*

Note. You may not *not* use the fact, first proved by Joseph Louis Lagrange in 1770, that any non-negative integer is a sum of four integer squares. Find a direct proof!

Hint. Suppose $n = 3\left(a^2 + b^2 + c^2\right)$, where a, b, and c are integers (some or all which might be equal to one another). You need to show that there are integers w, x, y, and z such that $n = w^2 + x^2 + y^2 + z^2$. One way to do this is to find suitable formulas for w, x, y, and z in terms of a, b, and c.*

Solution. The following is essentially Lewis Carroll's solution. Let

$$\begin{aligned} w &= a + b + c\,, \\ x &= b - c\,, \\ y &= c - a\,, \\ \text{and } z &= a - b\,. \end{aligned}$$

We can easily check that this works with a little algebra. Expand, cancel, and collect:

$$\begin{aligned} w^2 + x^2 + y^2 + z^2 &= (a+b+c)^2 + (b-c)^2 + (c-a)^2 + (a-b)^2 \\ &= \left(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\right) + \left(b^2 - 2bc + c^2\right) \\ &\quad + \left(c^2 - 2ca + a^2\right) + \left(a^2 - 2ab + b^2\right) \\ &= 3\left(a^2 + b^2 + c^2\right) \quad \blacksquare \end{aligned}$$

* This approach was used by Charles Lutwidge Dodgson (1832–1898), better known under his pen name of Lewis Carroll, to prove the result. Besides being a writer and poet of some renown – witness *Alice in Wonderland* and *Jabberwocky*, to name two of his better-known works – he was a mathematician, photographer, and inventor.